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EDITED BY
T. A. A. BROADBENT, M.A.
2 Buxton Avenue, Caversham, Reading

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THE MATHEMATICAL ASSOCIATION.

The Annual Meeting of the Mathematical Association was held at the Institute of Education on 4th-5th January, 1934.

On Thursday, 4th January, the proceedings opened at 2.15 p.m. with the transaction of business; the President, Professor G. N. Watson, was in the chair. The Report of the Council * for the year 1933 was presented and adopted, and the Treasurer's statement of the financial position received. On the nomination of the Council, Professor E. H. Neville was elected President for the year 1934, and Professor Watson was elected a Vice-President. Mr. W. Hope-Jones and Mr. C. J. A. Trimble retired from the Council and were not eligible for re-election; Mr. C. T. Daltry and Miss M. J. Griffith were elected in their places.

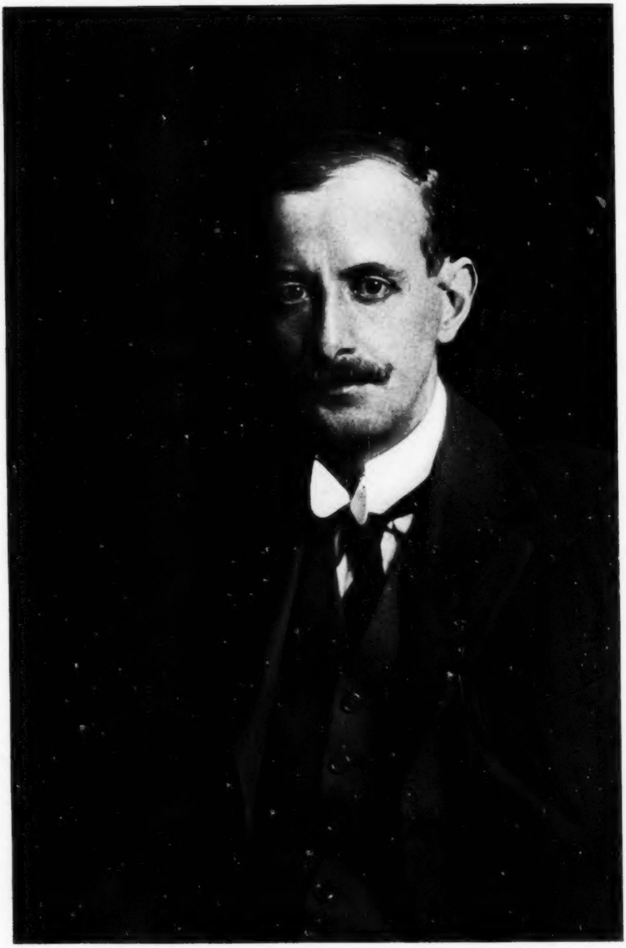
A decision on the alteration of Rule 13, proposed by Professor E. H. Neville, concerning the length of tenure of the office of President, was postponed till the next Annual Meeting.

Professor Watson then delivered his Presidential address: *Scraps from some Mathematical Note-Books.*† This was followed by Mr. R. A. M. Kearney's lecture, *Some Results of Relativity without the Theory of Tensors.*‡

On Friday, 5th January, at 10 a.m., Professor H. R. Hamley gave a paper on *The Function Concept in School Mathematics* ‡; this was followed by Mr. N. M. Gibbins' paper *The Eternal Triangle* ‡ and a discussion on *Mathematics in Central Schools* ‡ opened by Mr. G. T. Clark. The afternoon meeting began with Mr. T. A. A. Broadbent's paper on *Cauchy and the Complex Variable*; Dr. E. H. Askwith then spoke on *The Place of Solid Geometry in Elementary Teaching; Should not Space come before the Plane?* The final item was a discussion on *The Teaching of Differentials* ‡ opened by Professor G. Temple.

A Publishers' Exhibition was open during the two days.

* Pp. 2-4. † Pp. 5-18. ‡ To be published later.



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GEORGE NEVILLE WATSON, Sc.D., F.R.S.
President, January 1932—January 1934

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REPORT OF THE COUNCIL FOR THE YEAR 1933.

DURING the year 1933, 104 new members have been admitted to the Association. The number of members now on the roll is 1312, of whom 3 are Honorary members, 87 are life members by composition, 11 are life members under the old rule, and 1211 are ordinary members.

The Council regrets to have to announce the deaths of the following members of the Association: Mr. A. S. Grant, Principal J. L. S. Hatton, Professor E. W. Hobson, Mr. E. M. Langley, Mr. L. J. Rogers, Mr. F. H. Stevens, Miss M. T. Taylor, Professor W. C. Unwin, Mr. F. W. de Velling and Miss K. Watson.

Mr. A. S. Grant was a member of the Boys' Schools' Committee. Principal Hatton had been a member of the Association since 1902. He had been Principal of East London College for many years, and shortly before his death had been elected Vice-Chancellor of the University of London. Professor Hobson, Sadleirian Professor of Pure Mathematics in the University of Cambridge, was an Honorary member of the Association and held the office of President during the years 1911 and 1912. On many occasions, both before and after his Presidency, he did great service for the Teaching Committees, showing a wonderful understanding of the problems that confront teachers in schools. Mr. E. M. Langley was an Honorary Secretary of the Association from 1885 to 1893 and was Editor of the *Mathematical Gazette* during its first year, 1894. Mr. L. J. Rogers was Professor of Mathematics at Leeds from 1888 to 1919. Professor W. C. Unwin, Emeritus Professor of Engineering at the City and Guilds of London Institute, had been a member of the Association for 52 years, and Mr. F. H. Stevens, formerly of Clifton College, had been a member for 56 years. Mr. de Velling was formerly Headmaster of the Boulevard Municipal Secondary School, Hull.

The Mathematical Gazette.

The Index to Vols. I-XV of the *Gazette* was published in May and was distributed to members with the May number.

The Library.

Although there have been no large donations to the Library during the year, there has been a considerable number of small gifts. Very much more use has been made of the Library than in any previous year.

The Teaching Committees.

The **General Teaching Committee** has met only to deal with some details of correspondence and to consider the *Report on the Teaching of Algebra in Schools* put forward by the Boys' Schools' Committee. This Report has been approved for publication and is now in the press.

The **Boys' Schools' Committee** has completed during the year the Report referred to above, and is now considering the preparation of

a Report on Geometry to supplement the Report on this subject which was issued in 1923.

The Girls' Schools' Committee has held two meetings during the year 1933. At the first meeting the preparations for reprinting the 1929 edition of the *Report on the Teaching of Elementary Mathematics in Girls' Schools* were completed, some minor alterations being made. At the second meeting various University Entrance and Scholarship papers were discussed.

The Problem Bureau.

A considerable number of the problems submitted during the past year have come from members who have applied for help for the first time, so it would seem that the advantages of the Bureau are gradually becoming known to a wider circle.

The Branches.

The topic of general interest among the Branches this year has been the formation of the Branches Committee. It is hoped that by means of this Committee the Branches will be brought into closer relationship with the parent Association and a feeling of fellowship be fostered among the Branches themselves. The membership of every Branch shows an upward tendency; the attendance at the meetings and the interest taken in the discussions have been encouraging.

At a meeting of the London Branch an audience of about 110 persons met to hear an address from the President (Professor Cyril Burt) on "Intelligence Tests by, and for, Mathematicians". One meeting was devoted entirely to "Members' Topics".

The Yorkshire Branch has issued its annual report as usual with a full account of its proceedings. Professor S. Brodetsky initiated the members into the mysteries of "Yo-Yo", with practical illustrations by an expert.

At a meeting of the Manchester Branch Mr. Doughty read a paper on some "*New* Examination Methods" which was most illuminating. On 8th February a joint meeting with the University Society was held and the President of the Association (Professor G. N. Watson) read an inspiring paper on "Reciprocal Functions".

The Bristol Branch has held four meetings at which all the papers aroused great interest.

The President of the Midland Branch (Mr. Pratt) spoke on "The Human Touch in the Teaching of Mathematics". Other papers have been of a topological and astronomical character.

The North-Eastern Branch has been particularly fortunate in having a paper from Sir Westcott Abell on "Numerical Integration and Ship Calculations".

Many interesting papers have been read at the meetings of the Liverpool Branch. The President (Miss Ralph) took as her subject, "Why do we teach Mathematics?"

The Cardiff Branch has held three meetings during the session 1932-33. Addresses were given by the President of the Branch on "Geometry as a Branch of Invariant Theory," and by Mr. N. J. Loveridge, and Professor G. H. Livens. Members of the Branch were invited to hear the Selby Lecture given this year by Professor E. T. Whittaker.

At meetings of the South-West Wales Branch papers have been read by Professor A. R. Richardson on "The Mathematical Specialist in the School", and by the Rev. T. R. Ward Hill on "The Teaching of Elementary Geometry".

The membership of the Branches is made up of members and associates respectively as follows: London, 180, 90; Yorkshire, 52, 93; Manchester, 38, 65; Bristol, 13, 22; Midland, 29, 30; Liverpool, 17, 52; South-West Wales, 8, 32; North-Eastern, 58, 9; Southampton, 2, 24; Cardiff, 21, 32.

The Council.

Professor G. N. Watson, F.R.S., now retires from the office of President after serving the Association in that capacity for two years, and the Council desires to express its high and cordial appreciation of the services which he has rendered to the Association both personally and officially during that time. The Council nominates Professor E. H. Neville, Professor of Mathematics in the University of Reading, for election as President of the Association for the year 1934.

The Council nominates Professor Watson for election as a Vice-President of the Association.

The Council desires also to express the cordial thanks of the Association to Mr. T. A. A. Broadbent, Professor E. H. Neville and Mr. A. S. Gosset Tanner for the invaluable services which they continue to render to the Association as Editor, Librarian and Director of the Problem Bureau respectively; to Mr. F. Beames for his help in the management of the Library; and to Mr. W. Hope-Jones and Mr. C. J. A. Trimble, who now retire from the Council in their turn, for their unfailing assistance as members of Council since the years 1929 and 1930 respectively.

GLEANINGS FAR AND NEAR.

951. My advice would, therefore, be: if you must spoon-feed the very young, do so; but when they have shown a taste for the subject and attained the standard necessary for passing honours examinations, let them *then* be introduced to Euclid in his original form as an antidote to the more or less feeble echoes of him that are to be found in the ordinary school text-books of "geometry". I should be surprised if such qualified readers, making the acquaintance of Euclid for the first time, did not find it fascinating, a book to be read in bed or on a holiday, a book as difficult as any detective story to lay down when once begun. I know of one actual case, that of an undergraduate at Cambridge suddenly presented with a copy of Euclid, where this happened.—Sir Thomas Heath, Introduction to *The Elements of Euclid*, Everyman's Library edition, 1933, p. viii.

SCRAPS FROM SOME MATHEMATICAL NOTE-BOOKS.

Presidential Address to the Mathematical Association, 1934.

BY G. N. WATSON, SC.D., F.R.S.

IT seems to me that, no matter what doubts and perplexities may have assailed me in the selection of the subject and the contents of this Address, my primary duty on this occasion is obvious and inevitable; it is to express my appreciation of the impartiality of the Council in so arranging this afternoon's proceedings that my Address is subsequent to the discussion on Professor Neville's motion, so that supporters of the motion were unable to use the Address as an argument in favour of limiting each President to one year of office.

Before starting on my main topic, I propose to say a few words on the process of evolution (or, perhaps, I should say elimination) by which my initial intentions attained the form in which I present them to you. In my undergraduate days, now slightly more than a quarter of a century distant, the order of merit in the Mathematical Tripos still existed though its abolition had been decided upon; and, in certain circles, the discussion of the form of candidates and the places which they were likely to obtain in the next Tripos was not without interest; though the contest was, I think, free from any betting element, some impression of the general attitude to the examination may be gained by comparing it with the attitude of the sporting press to a forthcoming race. I have always got the impression from my father (who was a contemporary of Professor Forsyth) that in his time, a quarter of a century before mine, this interest in Tripos results as sporting events was considerably stronger. And if one goes back yet another quarter of a century, and turns over the pages of Bristed's *Five Years in an English University* (1851) or Everett's *On the Cam* (1865), the prominence and the degrading character of "Tripos shop" are somewhat startling.

It was probably with some thoughts of the order of merit in the Mathematical Tripos in mind that Professor Forsyth a few years ago, in an obituary notice of a mathematician of a previous generation, wrote as follows: "There is one activity in human nature which exercises a perennial lure for living minds. When a worker of recognized distinction in any field has completed his contribution to thought, some survivors delight in assigning him his place in an ordered hierarchy of memorable names. The task demands an easy omniscience which shall gauge all knowledge and all intellect, if its estimate of precedence in relative merit is to be promulgated with authority and received with belief". It occurred to me that a pleasant hour might be spent in the attempt to construct such a hierarchy of pre-eminent mathematicians and to give an account of some of their discoveries; but I soon realized that if I was to say anything worth saying about the work of perhaps a dozen mathematicians, I should require not a single lecture but a whole

course of lectures ; further, the apparently essential qualification of omniscience is lacking, because I think it unlikely that I am one of those about whom it can be said as it was said of Whewell "Science was his strength and omniscience his weakness". My aim consequently became more modest, and any regrets which I may have felt were allayed by the publication of the first volume of Prasad's *Some Great Mathematicians of the Nineteenth Century* which contains biographies of Gauss, Cauchy, Abel, Jacobi, Riemann and Weierstrass ; Prasad regards these as the first six among the greatest mathematicians of that century, and he intends to devote the second and third volumes of his work (it is permissible to share his hopes that they may be published in the very near future) to those whom he regards as next in order of eminence. There is no need for me to repeat here any of the criticisms of the first volume of the book which were recently made by a reviewer in the *Gazette* ; it is sufficient to express the hope that his high commendation may overbalance the effect of his adverse criticism, with the result that this stimulating book may be read as widely as it deserves to be.

My aim has thus dwindled until the upshot is that I propose to confine myself, not merely to the work of a single mathematician, but only to certain aspects of his work. The mathematician in question is Carl Friedrich Gauss, born at Braunschweig, 30th April, 1777, died at Göttingen, 23rd February, 1855. On 30th March, 1796, a month before the completion of his nineteenth year, a day described by Klein (with, I think, some exaggeration) as "sein Tag von Damaskus", Gauss started entering his discoveries as he made them in a small volume, now usually described as his "Tagebuch", though he himself in a letter to Olbers calls it a "Notizen-Journal". The Tagebuch contains 146 statements of discoveries and enunciations of theorems, occupying nineteen pages ; most of these statements are dated, and it appears that as many as 121 of them were entered within six years of the date of the first entry, *i.e.* before October, 1801 ; the last entry bears the date 9th July, 1814. After the death of Gauss the Tagebuch remained in the possession of his family for many years ; it was published with annotations by Klein in 1901 in the *Festschrift zur Feier des hundertfünfzigjährigen Bestehens der Königlichen Gesellschaft der Wissenschaften zu Göttingen*. It has since been republished with a facsimile in the tenth volume of *C. F. Gauss Werke*.

Notwithstanding this dual publication, the contents of the Tagebuch are probably not widely known in this country ; and it seems worth while to make a small selection of the most striking results contained therein and to describe their developments during the last 130 years ; on some of these results the last word has not yet been said. Unfortunately the time at my disposal does not admit of my quoting at length from the Tagebuch ; were I able to do so, my quotations would render evident the greatness of the enthusiasm and exuberance which Gauss so carefully concealed in the severe austerity of his published work.

The first entry in the Tagebuch is as follows:

Principia quibus innititur sectio circuli, ac divisibilitas eiusdem geometrica in septemdecim partes, etc.

[1796] Mart. 30. Brunsv[igae].

This discovery of the possibility of dividing the circumference of a circle into 17 equal parts is amplified in a communication to the *Intelligenzblatt der allgemeinen Litteraturzeitung* of 1st June, 1796; the best way in which I can describe the significance of the discovery is to give a free translation of the letter.

"Every student of Geometry knows that various ordinary (regular) polygons, namely the triangle, pentagon, quindecagon, and those which arise from repeated duplication of the number of the sides of these polygons admit of being constructed by geometrical methods. One had got so far as early as the time of Euclid, and it seems to have been the general conviction that the domain of elementary Geometry extends no further; at least I know of no successful attempt to extend its limits on this side.

"So much the more, I think, does the discovery deserve attention that, in addition to these ordinary polygons, another set is capable of geometrical construction, for example the (regular) 17-gon. This discovery is, in fact, only a corollary of a theory of great extent which is still not quite complete, and this will be laid before the public as soon as its completion has been obtained."

Before I give an account of the general theory, I take the special case of the 17-gon. It is easy to see that a geometrical construction by straight lines and circles is possible if it is possible to express $\cos 2\pi/17$ and $\sin 2\pi/17$ in terms of radicals, all the radical signs being square roots only. Since

$$\cos\frac{2\pi}{17}+i\sin\frac{2\pi}{17}=e^{2\pi i/17},$$

it is thus necessary to solve the equation

$$x^{17}=e^{2\pi i}=1 \quad (x\neq 1)$$

by radicals. Let x first denote any complex root of this equation, so that, when the factor $x-1$ is omitted,

$$x^{16}+x^{15}+x^{14}+\ldots+x+1=0;$$

and write

$$x+x^9+x^{13}+x^{15}+x^{16}+x^8+x^4+x^2=U_1,$$
$$x^3+x^{10}+x^5+x^{11}+x^{14}+x^7+x^{12}+x^6=U_2;$$

it is easy to verify that

$$U_1+U_2=-1, \quad U_1U_2=-4,$$

so that U_1 and U_2 are the roots of the quadratic equation

$$U^2+U-4=0,$$

and $$U_1, U_2=\tfrac{1}{2}\{-1\pm\sqrt{17}\}.$$

Next write

$$x+x^{13}+x^{16}+x^{4}=V_1,$$
$$x^{9}+x^{15}+x^{8}+x^{2}=V_2,$$
$$x^{3}+x^{5}+x^{14}+x^{12}=V_3,$$
$$x^{10}+x^{11}+x^{7}+x^{6}=V_4,$$

so that

$$V_1+V_2=U_1, \quad V_1V_2=-1,$$
$$V_3+V_4=U_2, \quad V_3V_4=-1;$$

and hence V_1, V_2, V_3, V_4 are the roots of quadratic equations whose coefficients involve U_1 and U_2, and so they are ultimately expressible by radicals. Finally, let

$$x+x^{16}=W_1, \quad x^{13}+x^{4}=W_2,$$

. . . .

so that

$$W_1+W_2=V_1, \quad W_1W_2=V_3,$$

and hence W_1 is expressible by radicals. When we take x to be the special root $e^{2\pi i/17}$, we have $W_1=2\cos 2\pi/17$.

When x is given this special value, it is easy to fix the ambiguities of sign which have arisen in the solutions of the various quadratic equations, and so an expression for $2\cos 2\pi/17$ has been obtained involving quadratic irrationalities only. The construction of the regular 17-gon by ruler and compasses is now a perfectly straightforward matter; such constructions have been given by a number of writers, starting with Gauss himself, and I do not propose to discuss them apart from mentioning that a particularly elegant construction has been given by H. W. Richmond.*

I now come to the more general theorem, to the effect that a regular polygon of p sides admits of construction by ruler and compasses if p is a prime number of the form 2^m+1, and, as a consequence of this, that a regular polygon of $2^r p_1 p_2 p_3 \ldots$ sides admits of construction by ruler and compasses if r is any positive integer (zero included) and p_1, p_2, p_3, ... are *different* primes of the form 2^m+1.

The second part of the theorem is derivable from the first part in precisely the way in which Euclid derives the construction of the quindecagon from the constructions for the triangle and pentagon.

The first part of the theorem necessitates a knowledge of the primality of numbers of the form 2^m+1; it is obvious that, when m is odd, 2^m+1 is divisible by $2+1$, *i.e.* by 3, and so it is not prime unless $m=1$. Again, if m has any odd factor, q say, it is evident that 2^m+1 is divisible by $2^{m/q}+1$. It follows that the only values of m for which it is possible for 2^m+1 to be prime are unity and powers of 2. The five smallest relevant values of 2^m+1 are therefore 3, 5, 17, 257 and 65537, these being all primes. There is no need for me to say more about the first three of these, and I come to the number 257; the determination of $\cos 2\pi/257$ in terms of quadratic irrationalities is evidently a much more tedious

* *Math. Annalen*, 67 (1909), 459-461.

business than the corresponding problem for $\cos 2\pi/17$. It has, however, been effected by Friedrich Julius Richelot,* who was born at Königsberg on 6th November, 1808, and died on 31st March, 1875. It might be anticipated that life would not be long enough to deal with the determination of $\cos 2\pi/65537$, but the anticipation would be incorrect. The problem has been attacked by Johann Gustav Hermes, who was born on 20th June, 1846, also at Königsberg, and died on 8th June, 1912. Hermes took the degree of D.Ph. at Königsberg in 1878 and later was Professor at the Gymnasium Georgianum at Lingen. He published a pamphlet of a dozen pages on his problem at Königsberg in 1889, while an epitome of his final results was published † in the *Göttinger Nachrichten* five years later.

I now make some remarks on the sequence of numbers 5, 17, 257, 65537, ... It was conjectured by Fermat in 1640 that they are all prime, but the conjecture is incorrect; in consequence of the conjecture they are usually known as Fermat's numbers and denoted by $F_1, F_2, F_3, F_4, \ldots$; an account of all that was known about them down to 1919 is the subject of Chap. XV of Dickson's *History of the Theory of Numbers*. As has been stated, F_1, F_2, F_3, F_4 are prime, but it was discovered by Euler that F_5 is composite, and, in fact, that

$$F_5 = 2^{32} + 1 = 641 \times 67\,00417.$$

It is stated by Dickson that F_n is composite for the following values of n:

$$5, 6, 7, 8, 9, 11, 12, 18, 23, 36, 38, 73;$$

while nothing is known about the primality of F_n for all other values of n exceeding 4. It thus appears that anyone who wishes to surpass Hermes will have to deal with the construction of the regular p-gon, where, in the most favourable circumstances, p is equal to $2^{1024}+1$, while it is by no means impossible that the smallest available value of p is either $2^{8192}+1$ or some much larger number.

I mention in passing that it was shown by F. Landry at the age of eighty-two that F_6 is the product of the two primes

$$2\,74177, \quad 6728\,04213\,10721,$$

and that it was shown by J. C. Morehead in 1906 that F_{73} has the prime factor $5 \cdot 2^{75}+1$; and I must refer those who want further information about these numbers to Dickson's *History*.

Enough has now been said about the first entry in the Tagebuch, and I turn to the second entry:

Numerorum primorum non omnes numeros infra ipsos residua quadratica esse posse demonstratione munitum.

[1796] Apr. 8. Ibid. [Brunsvigae].

This entry refers to his first proof of the quadratic-reciprocity theorem, frequently described as "theorema aureum"; the theorem

* *Journal für Math.* 9 (1832), 1-26.

† *Gött. Nach.* 1894, 170-186.

was discovered by Gauss by induction in March, 1795, though it had been discovered and proved earlier by Euler. The Tagebuch mentions the other five proofs discovered by Gauss, of which I quote the entry (16) on the second proof:

Nova theorematis aurei demonstratio a priori toto caelo diversa eaque haud parum elegans. [1796] 27. Iun.

An adjacent entry (18) shows signs of delight in a discovery:

ΕΥΡΗΚΑ ! num[erus] $=\Delta+\Delta+\Delta$.

[1796] 10. Iul. Gott[ingae].

This is a statement of the theorem that every integer is expressible as the sum of three triangular numbers, *i.e.* numbers of the form $\frac{1}{2}q(q+1)$. If

$$m=\sum_{r=1}^{3}\tfrac{1}{2}q_r(q_r+1),$$

then

$$8m+3=\sum_{r=1}^{3}(2q_r+1)^2;$$

and so the statement is equivalent to the theorem that every integer of the form $8m+3$ is expressible as the sum of three odd squares. In all probability this entry is connected with the theorem which Gauss proves in his *Disquisitiones Arithmeticae* that the number of representations of $8m+3$ as the sum of three squares is equal to

$$2^{\mu+2}h,$$

where μ is the number of distinct prime factors of $8m+3$ and h is the number of classes of binary quadratic forms of determinant $-(8m+3)$. Any account of the theory of class-numbers would take me too far afield, and all that need be said here is that

$$x^2+xy+(2m+1)y^2$$

is a binary quadratic form with the requisite determinant (*i.e.* there must be at least one class of such forms, the class containing at least one member), so that h is not zero, but is at least equal to unity. Hence any positive integer is expressible in at least one way as the sum of three triangular numbers.

A few words about the expression of an integer as a sum of squares may not be out of place. Not every integer is expressible as the sum of two squares; it follows from a theorem proved by Jacobi that the number of representations of a positive integer n as the sum of two squares is

$$4(d-d'),$$

where d and d' are the number of divisors of n which are of the form $4m+1$ and the number which are of the form $4m+3$ respectively; this result, implicit in Jacobi's work, was first stated explicitly by Dirichlet.

It was inferred by Bachet as early as 1621 that *every* positive integer is expressible as the sum of four squares, though he was

unable to prove it. The simplest proof is obtainable from a certain expansion, also due to Jacobi, which makes it evident that the number of representations of n as the sum of four squares is obtained by writing n in the form $2^a q$ where q is odd; if $\sigma(q)$ denotes the sum of the divisors of q, the required number of representations is $8\sigma(q)$ or $24\sigma(q)$ according as $\alpha=0$ or $\alpha>0$.

There are similar theorems for the number of representations of n as the sum of 5, 6, 7, ... squares into which I will not enter.

Any positive integer n, which is not of the form $2^a(8m+7)$ is expressible as the sum of three squares, and numbers of the excepted form are not so expressible; the number of representations was discovered by Gauss in various forms involving class-numbers, of which the form which I have quoted for integers of the form $8m+3$ is typical.

There is an extremely fascinating problem arising out of these representations, namely, is there any approximate formula for the number of representations $\rho_s(n)$ of a positive integer n as the sum of s squares when n is large?

For values of $s \geqslant 4$, the last word on the subject has probably been said, since the answer given by Hardy * in 1920 seems adequate and final. As an example of the results in his paper, I quote the formula for $s=5$ when n has no square factor and $n-1$ is not divisible by 4, namely,

$$\rho_5(n)=\frac{160}{\pi^2}\, n^{\frac{3}{2}} \sum \left(\frac{n}{m}\right)\frac{1}{m^2},$$

where m runs through all values prime to n, and (n/m) is the "quadratic-reciprocity symbol of Legendre and Jacobi" which is equal to $+1$ or to -1 according to the relative values of n and m.

For $s=2$, the question before us is pointless; for this value of s, the natural question of a cognate character is "What approximate formula is there for the number of integers less than a large number x which are expressible as the sum of two squares?" The answer to this question has been supplied by Landau,† who has obtained the approximate formula

$$\frac{Cx}{\sqrt{\log x}},$$

where

$$C=\left\{2\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{7^2}\right)\left(1-\frac{1}{11^2}\right)\left(1-\frac{1}{19^2}\right)\dots\right\}^{-\frac{1}{2}}=0{\cdot}764\dots.$$

The most amazing thing about this formula is that it was discovered, apparently independently, by Ramanujan in his early days in India, and it appears in its appropriate place in his manuscript note-books.

I have so far been silent about the case $s=3$; if one considers larger values for s, one is led to anticipate that $\rho_3(n)$ would be, roughly speaking, of the order of magnitude of $\sqrt{n}$ when n tends to infinity either through all integral values, with the exception of

* *Trans. American Math. Soc.* 21 (1920), 255-284.

† *Archiv der Math. und Phys.* (3), 13 (1908), 305-312.

those of the form $2^a(8m+7)$, or perhaps through a certain sequence of integral values ; but, unfortunately, all attempts to prove any such result have hitherto failed, and all that is known consists of theorems about what may be called the *average order* of magnitude of $\rho_3(n)$. This failure is much to be regretted because of the importance of $\rho_3(n)$ in the theory of the number of classes of binary quadratic forms of determinant $-n$.

I turn now to the results in the Tagebuch concerning continued fractions ; there are two entries on the subject, the first (7) consisting of two assertions of the equality of a continued fraction and a series, namely,

$$1-2+8-64+\dots$$
$$=\frac{1}{1+}\frac{2}{1+}\frac{2}{1+}\frac{8}{1+}\frac{12}{1+}\frac{32}{1+}\frac{56}{1+}\frac{128}{1+}\cdots,$$
$$1-1+1\,.\,3-1\,.\,3\,.\,7+1\,.\,3\,.\,7\,.\,16-\dots$$
$$=\frac{1}{1+}\frac{1}{1+}\frac{2}{1+}\frac{6}{1+}\frac{12}{1+}\frac{28}{1+}\cdots.$$

[1796] Mai. 24. G[ottingae].

These assertions are both false. In each case the series is not convergent, but that in itself would not be a matter of great importance ; thanks to the researches of Stieltjes which were carried out about a half a century ago, a good deal is now known about the transformations of continued fractions into series, and there are certain classes of convergent continued fractions for which the transformations lead to series which are not convergent, but which are of the type usually described as asymptotic ; and, when the sums of these series are determined by any of the standard methods which are used to define the sum of an asymptotic series, the results are equal to the corresponding continued fractions.

This happy state of affairs does not hold with either of the continued fractions given by Gauss ; it is not possible to assign a sum to either of the series concerned, and neither of the continued fractions is convergent. The continued fractions are oscillatory, and it would be interesting to attack the unsolved (but probably not unduly difficult) problem of determining the two 'limits of oscillation'.

This unfortunate mistake was, to some extent, rectified by entry 58 in the Tagebuch, which reads thus :

Amplificatio prop[ositionis] penult[imae] p[aginae] 1, scilicet

$$1-a+a^3-a^6+a^{10}-\dots$$
$$=\frac{1}{1+}\frac{a}{1+}\frac{a^2-a}{1}\frac{a^3}{+1+}\frac{a^4-a^2}{1}\frac{a^5}{+1+}\ \text{etc.}$$

Unde facile omnes series, ubi exp[onentes] ser[ierum] sec[undi] ordinis constituunt, transformantur. [1797] Febr. 16.

The more general result,

$$1+qx+q^4x^2+q^9x^3+q^{16}x^4+\dots$$
$$=\frac{1}{1-}\frac{qx}{1+}\frac{q(1-q^2)x}{1}\frac{q^5x}{-1+}\frac{q^3(1-q^4)x}{1}\frac{q^9x}{-1+}\frac{q^5(1-q^6)x}{1}\frac{q^{13}x}{-1+}\cdots,$$

was contained in a paper dated December, 1843, and published in 1844 by Eisenstein,* who is known to have visited Gauss at Göttingen in July of the latter year; a proof of the more general result was published by Heine† in 1846.

The first of the Gaussian formulae is obtainable by putting $q=\sqrt{2}$, $x=-\sqrt{2}$ in Eisenstein's formula, while the second is similarly obtainable by putting $q=2$, $x=1$ in the formula

$$1+(1-q)x+(1-q)(1-q^2)x^2+(1-q)(1-q^2)(1-q^3)x^3+\ldots$$
$$=\frac{1}{1-}\,\frac{(1-q)x}{1}\,\frac{q(1-q)x}{1-}\,\frac{q(1-q^2)x}{1-}\,\frac{q^2(1-q^2)x}{1-}\,\frac{q^2(1-q^3)x}{1-}\cdots,$$

which is a special case of another result proved by Heine.

The reason for the incorrectness of the numerical results stated by Gauss is now apparent; the expressions which occur in Heine's formulae, regarded as functions of a complex variable q, exist only inside the circle $|q|=1$, this circle being a line of singularities, or a natural barrier as it is sometimes called, of the functions concerned. The whole theory of analytic continuation and all the methods of defining the sum of an asymptotic series are incapable of breaking through such a barrier in such a way as to make it possible to attach a meaning to the various functions involved when q is assigned a value whose modulus exceeds unity.

At this point I ought to leave the subject of continued fractions, but I am going to allow the fascination which the subject exerts over me (far be it from me to compare this fascination to that which a snake exerts over a rabbit) to lead me into being irrelevant for a moment. I take it that all who are similarly attracted are familiar with the admirable work *Die Lehre von den Kettenbrüchen* by Oskar Perron, the second edition of which gives a complete account of everything published on the subject before 1929. With that assumption in mind I am going to state a theorem which is not contained in Perron's book. It is one of the most beautiful of the many beautiful theorems in Ramanujan's unpublished note-books. The theorem is as follows:

Let $P=\prod\Gamma\left(\frac{\alpha\pm\beta\pm\gamma\pm\delta\pm\epsilon+1}{2}\right)$ (0, 2 or 4 minus signs),

and $Q=\prod\Gamma\left(\frac{\alpha\pm\beta\pm\gamma\pm\delta\pm\epsilon+1}{2}\right)$ (1 or 3 minus signs),

with eight gamma functions in each product. Then

$$\frac{P-Q}{P+Q}=\frac{8\alpha\beta\gamma\delta\epsilon}{\{2(\Sigma\alpha^4+1)-(\Sigma\alpha^2-1)^2-2^2\}+}\,\frac{64\Pi(\alpha^2-1^2)}{3\{2(\Sigma\alpha^4+1)-(\Sigma\alpha^2-5)^2-6^2\}+}\,\frac{64\Pi(\alpha^2-2^2)}{5\{2(\Sigma\alpha^4+1)-(\Sigma\alpha^2-9)^2-10^2\}+}\cdots,$$

provided that one of the numbers α, β, γ, δ, ϵ is an integer.

* *Journal für Math.* 27 (1844), 75-79.

† *Journal für Math.* 32 (1846), 210-212; see also *ibid.* 34 (1847), 285-328.

It is characteristic of Ramanujan, as would be expected by those who knew him, that he has left no indication as to how he either discovered or proved this theorem. Notwithstanding the complexity and generality of the theorem, there exists a proof of it which I regard as extremely simple and extremely beautiful ; and I do not think that I am being unduly immodest in describing the proof in these terms. There is another theorem, closely related to this, but with one more degree of generality, which was not stated by Ramanujan, and my method of proving the special theorem is also effective in proving the more general theorem.

I come now to the completely different topic raised by the entry numbered 82 in the Tagebuch, which runs as follows :

Seriei

$$x-\tfrac{1}{2}x^2+\tfrac{1}{12}x^3-\tfrac{1}{144}x^4\ldots$$

summam consideravimus inveniamusque eam $=0$, si

$$2\sqrt{x}+\frac{3}{16}\frac{1}{\sqrt{x}}-\frac{21}{1024}\frac{1}{\sqrt{x^3}}\ldots=(k+\tfrac{1}{4})\pi.$$

Brunsv[igae, 1797] Oct. 16.

The form of the result is immediately suggestive of Stokes' formula for the zeros of a Bessel function, and we have, in fact, in the ordinary notation of Bessel functions,

$$J_1(2\sqrt{x})\sqrt{x}=x-\frac{x^2}{2}+\frac{x^3}{12}-\frac{x^4}{144}+\ldots.$$

The formula given by Stokes half a century after the date of this entry is derived from the asymptotic expansion of the Bessel function, and I think that I am correct in saying that no asymptotic expansions of Bessel functions were published before Poisson dealt with $J_0(z)$ in 1823. The discovery made by Gauss a quarter of a century before Poisson and half a century before Stokes affords some indication of the pre-eminence of Gauss as a mathematician ; one may ask why Gauss never followed up this result (so far as I know, he never did anything else worth mention on the subject of Bessel functions, notwithstanding their obvious attractions to an astronomer). I think that the answer to this question is to be found in his dominant characteristic of never building on an insecure foundation. In his time there was no rigorous theory of asymptotic series ; and, without doubting his capacity to construct such a theory, one can easily imagine that the requisite type of analysis lies in a field in which he was not really interested ; and so he left on one side a branch of analysis in which Stokes worked fearlessly because he considered that the results justified the means used to obtain them, and modern mathematicians work fearlessly because, thanks to Poincaré, Stieltjes and Borel, they have an adequate theory to rely upon.

As an instance of an astronomical problem I quote entry 117 in the Tagebuch :

Iisdem diebus Pascha Judaeorum per methodum novam determinare docuimus. [1801] Apr. 1.

and, as a conundrum, I quote entry 43:

Vicimus GEGAN. [1796] Oct. 21. Bruns[vigae].

It is not known what problem this describes, and we are similarly ignorant of the meaning of the interpolation "REV. GALEN" in an entry on parallax numbered 97; to judge from a note made elsewhere, the latter may be connected with his work on the Arithmetic-Geometric Mean.

On the Arithmetic-Geometric Mean there are five entries, and on elliptic functions generally there are a couple of dozen more; it is obvious that the least unsatisfactory way of dealing with such a vast subject in an hour's lecture is to omit it altogether, and so I come to the last topic which I propose to discuss, namely, the zeta function of Riemann coupled with the "Prime number theorem".

The zeta function, defined by the formula

$$\zeta(s)=\frac{1}{1^s}+\frac{1}{2^s}+\frac{1}{3^s}+\dots,$$

appears in a slightly disguised form in entry 45:

Incepi Expressionem

$$1-\frac{1}{2^\omega}+\frac{1}{3^\omega}\dots$$

in seriem transmutare secundum potestates ipsius ω progredientem.
[1796] Nov. 26. G[ottingae].

It is a fairly simple matter to prove that this series is connected with the zeta function by the relation

$$\frac{1}{1^\omega}-\frac{1}{2^\omega}+\frac{1}{3^\omega}-\dots=\left(1-\frac{2}{2^\omega}\right)\zeta(\omega).$$

The prime number theorem is not mentioned in the Tagebuch; but in a copy of Schulze's logarithm tables which Gauss has inscribed "Gauss. 1791" he has made an entry:

Primzahlen unter $a(=\infty)$,

$$\frac{a}{la}.$$

which one can readily interpret as the statement that the number of primes less than a large number x is approximately equal to $x/\log_e x$.

Probably everybody who reads this Address has listened at one time or another to a lecture on the prime number theorem and its relations with the Riemann zeta function, and, in particular, with the celebrated hypothesis of Riemann that all the roots of the equation

$$\zeta(s)=0,$$

which are not real and negative, are of the form

$$\tfrac{1}{2}\pm it,$$

where t is a real positive number. At the risk of being tedious I feel it necessary to repeat certain familiar results, and I can only hope that the statement of the theorem to which they are going to lead may be regarded as an adequate compensation for this preliminary repetition. It will be familiar knowledge that nobody has yet succeeded in proving the Riemann hypothesis, and that fortunately the truth of the hypothesis is not an essential element in the proof of the prime number theorem. This theorem is now (following the work of Tchebychef * published in 1851) usually expressed in the form

$$\pi(x) \sim \operatorname{li} x,$$

where $\pi(x)$ denotes the number of primes less than the large number x, and $\operatorname{li} x$ denotes the 'logarithmic integral'

$$\operatorname{li} x = \int_0^x \frac{dt}{\log_e t},$$

to which $x/\log_e x$ is an approximation when x is large; the first rigorous proofs of the theorem were constructed by Hadamard † and de la Vallée Poussin ‡ as recently as the last decade of the nineteenth century.

A few years subsequent to Tchebychef, Riemann § in his epoch-making memoir "Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse" obtained a much more elaborate approximation, by methods which were not completely justified, namely,

$$\pi(x) \sim \operatorname{li} x - \tfrac{1}{2} \operatorname{li} x^{\frac{1}{2}} - \tfrac{1}{3} \operatorname{li} x^{\frac{1}{3}} - \tfrac{1}{5} \operatorname{li} x^{\frac{1}{5}} + \tfrac{1}{6} \operatorname{li} x^{\frac{1}{6}} - \dots .$$

The law of signs of the terms on the right is not quite obvious; the term in $\operatorname{li} x^{1/n}$ has a plus or minus sign attached according as the number of distinct ways of factorising n is even or odd, and the term is omitted when n has a square factor.

So far as numerical evidence goes, Riemann's elaborate formula would appear to give a decidedly better approximation to $\pi(x)$ than is given by the more simple formula of Tchebychef; I quote a table which illustrates this, the headings R. and T. indicating the values given by Riemann's and Tchebychef's formulae respectively (correct to the nearest integer):

x	$\pi(x)$	R.	T.
50000	5133	5133	5167
1 00000	9592	9587	9630
10 00000	78498	78528	78628
100 00000	6 64579	6 64667	6 64918
1000 00000	57 61455	57 61552	57 62209
10000 00000	508 47478	508 47455	508 49235

* Republished in *J. de Math.* (1), 17 (1852), 366-390.

† *Bull. de la Soc. math. de France*, 24 (1896), 199-220.

‡ *Ann. de la Soc. sci. de Bruxelles*, 20 (1896), 183-256.

§ *Berliner Monatsberichte*, 1859, 671-680.

Numerical evidence may, however, be deceptive when it is incomplete ; and in the problem under consideration, in the nature of things, it can never be complete. The inadequacy of the table just given to afford any indication of what happened for values of x beyond the range of the table was conclusively established when in 1914 it was proved by Littlewood * that, as x increases from 2 to infinity, the difference $\pi(x) - \mathrm{li}\, x$ changes sign an infinite number of times. Whenever $\pi(x) - \mathrm{li}\, x$ is positive, Riemann's expression (being less than $\mathrm{li}\, x$) is a *worse* approximation than Tchebychef's.

Unfortunately Littlewood's method does not supply numerical results ; his theorem is of the nature of an *existence-theorem* which proves that there is a smallest integral value of x, say x_0, for which $\pi(x) - \mathrm{li}\, x$ is positive, but it does not afford any means of determining x_0.

Let me give a very elementary illustration of this kind of deficiency of some existence-theorems. Consider a function $f(z)$, of a complex variable z, defined by a Taylor series

$$f(z) = a_0 + a_1 z + a_2 z^2 + \dots .$$

There is a simple existence-theorem to the effect that on the circumference of the circle of convergence of the series there exists at least one singularity of the function. To prove the theorem, one has merely to assume that there is no such singularity, so that in the proof of Taylor's expansion theorem by Cauchy's method of contour integration (given in all text-books on Analysis) the contour can be taken to be a circle concentric with the circle of convergence and *larger* than the circle of convergence ; and, by the proof just described, the series converges throughout the interior of the larger circle, and therefore at some points *outside* the circle of convergence. The contradiction so obtained shows that the hypothesis that no singularity on the circumference exists must be false.

This reasoning is simple enough ; but it obviously does not directly supply any means of determining the actual positions of the singularities of the function, as will be realized only too well by those who have found it necessary to work through the excellent little book by Hadamard *La série de Taylor et son prolongement analytique*, or the larger and more recent work by Dienes *The Taylor Series*.

The deficiency in the theorems used by Littlewood has quite recently been made good by S. Skewes,† who has thus been able to determine an upper limit for x_0. Skewes has not yet published his work in full, and at present he finds it necessary to assume the truth of the Riemann hypothesis ; oddly enough, while Littlewood's theorem that x_0 exists is more easily proved if the Riemann hypothesis is false (though Littlewood's theorem is valid whether the Riemann hypothesis is true or false), it appears to be decidedly

* *Comptes Rendus*, 158 (1914), 1869-1872.

† *Journal London Math. Soc.* 8 (1933), 277-283.

more difficult to obtain numerical results if the Riemann hypothesis is false than if it is true.

I now state the actual result obtained by Skewes :

$$x_0 < e^{e^{e^{79}}}, \quad x_0 < 10^{10^{10^{34}}}.$$

It is possible that this result may be improved at a subsequent date by using more elaborate methods. It is proverbially dangerous to make prophecies about unproved results in the analytic theory of numbers ; in view of my ignorance of the subject, it may even be rash for me to say that all that I will commit myself to is the statement that, if x_0 is ever calculated exactly, I should not be surprised if its value were obtained by replacing the number 79 by some number between 2 and 20, and that I should be rather less surprised if it had to be replaced by a number less than 2 than I should be if it had to be replaced by a number greater than 20.

I should like to remark that the number obtained by Skewes is really quite a large one. At the time of his Presidential Address three years ago my predecessor in this Chair stated that the number of protons in the universe is either 7×10^{78} or else 14×10^{78}, the ambiguity being due to the impossibility of determining whether each proton has been counted once or twice. It is not an unknown phenomenon for Physical Theories to undergo violent fluctuations, and I regret that my knowledge of Mathematical Physics is not sufficiently up-to-date for me to assert categorically that the number of protons in the universe is still miserably small compared with Skewes' upper limit for x_0.

My selection of topics suggested by entries in the Tagebuch is now complete ; it may be that it is convincing evidence that Gauss was one of the greatest mathematicians of all time, but that is a side issue and not the main object of my Address ; my real aim has been to stress the dictum of Abel that the way to make progress in Mathematics is to study the masters and not the pupils. If I have incited anybody to make himself more familiar with the work of the masters, whether by reading Prasad's book or by working through the two volumes of Klein's *Entwicklung der Mathematik im* 19. *Jahrhundert*, still more if I have cast over anybody the spell which the Tagebuch holds over me, this lecture will not have been delivered in vain. G. N. W.

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952. Word-numerals and Number-classes.

Quoting from Brahmagupta (6th cent. A.D.) in description of the old Indian word-numerals : " If you want to write one, express it by everything which is unique, as the earth, the moon ; two by everything which is double, as for example black and white ; three by everything which is threefold. . . ."

Krishnaswami comments :

Does not this bring us very near the modern definition of number, for, according to Bertrand Russell, " a number is any collection which is the number of one of its members, or more simply still, a number is anything which is the number of some class " ?—" Some glimpses of ancient Indian mathematics " ; *The Mathematics Student*, vol. 1, p. 6. [Per Prof. E. H. Neville.]

PANSYMMETRICAL PENCILS.

By B. M. Peek.

In Fig. 1, $AA_1A_2A_3$ ... is a rectangular hyperbola, whose asymptotes are OT, OT', and whose vertex is A. We make the following construction :

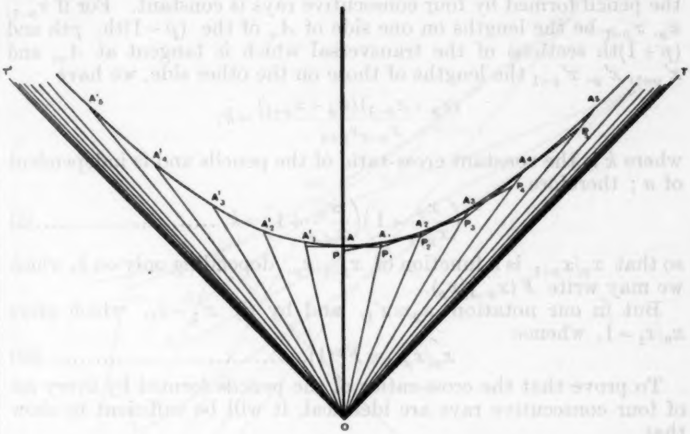

Fig. 1.

At A_1, a point on the hyperbola, a tangent is drawn, cutting OA at P. The tangent at A is drawn, cutting OA_1 at P_1. Let the other tangent from P_1 touch the hyperbola at A_2. Produce PA_1 to cut OA_2 at P_2. Let the other tangent from P_2 touch the hyperbola at A_3, and let P_1A_2 produced cut OA_3 at P_3.

Consider this process to be continued indefinitely and to be repeated on the other side of the axis OA, A'_1 being the point of contact of the other tangent from P. We obtain a pencil of rays OA_n, OA'_n which gradually lie closer together, becoming infinite in number as the asymptotes, which are the ultimate rays of the pencil, are approached. The system is obviously symmetrical about OA, and I shall show that this pencil has further remarkable properties, which I have named " pansymmetry ". These properties can be stated thus :

If a tangent to the hyperbola at the point A_n be produced both ways, it will be symmetrically divided by all the rays of the pencil, the pth section on the one side of A_n being equal to the pth section on the other side.

Furthermore, *the ratio of the pth to the $(p+1)$th section is the same, whatever integral value be taken for n*, the sections being counted in either direction outward from A_n, the two adjacent to A_n being numbered " 1 ".

Now it is easily deduced from well-known properties of the hyperbola that $PA_1 = A_1P_2$, $P_1A_2 = A_2P_3$, and generally

$$P_{m-1}A_m = A_mP_{m+1}, \quad \text{.........................(i)}$$

so that the first section on one side of A_n is in every case equal to the first section on the other side. Clearly, then, the properties of pansymmetry will hold, if it can be shown that the cross-ratio of the pencil formed by four consecutive rays is constant. For if x_{p-1}, x_p, x_{p+1} be the lengths on one side of A_n of the $(p-1)$th, pth and $(p+1)$th sections of the transversal which is tangent at A_n, and x'_{p-1}, x'_p, x'_{p+1} the lengths of those on the other side, we have

$$\frac{(x_p + x_{p-1})(x_p + x_{p+1})}{x_{p-1}x_{p+1}} = k,$$

where k is the constant cross-ratio of the pencils and is independent of n; therefore

$$\left(\frac{x_p}{x_{p+1}} + 1\right)\left(\frac{x_p}{x_{p-1}} + 1\right) = k, \quad \text{.........................(ii)}$$

so that x_p/x_{p+1} is a function of x_{p-1}/x_p, depending only on k, which we may write $F(x_{p-1}/x_p)$.

But in our notation $x_0 \equiv x'_1$, and by (i) $x'_1 = x_1$, which gives $x_0/x_1 = 1$, whence

$$x_p/x_{p+1} \equiv F^p(1). \quad \text{.................................(iii)}$$

To prove that the cross-ratios of the pencils formed by every set of four consecutive rays are identical, it will be sufficient to show that

$$O(A_{n-1}A_nA_{n+1}A_{n+2}) = O(A_{n-2}A_{n-1}A_nA_{n+1}),$$

for nowhere in the proof will it be assumed that the angle OAP_1 is a right-angle. The extrapolation may therefore be carried through A right round the hyperbola.

In Fig. 2, $OA_{n-2}, \ldots, OA_{n+2}$ are five consecutive rays. $P_{n-2}P_n$, the tangent at A_{n-1}, cuts OA_{n+1} at Q; $P_{n+2}P_n$, the tangent at A_{n+1}, cuts OA_{n-1} at R. Now $P_{n-2}A_{n-1} = A_{n-1}P_n$, and $P_{n+2}A_{n+1} = A_{n+1}P_n$, by (i). If therefore it can be shown that

$$A_{n-1}P_n : P_nQ = A_{n+1}P_n : P_nR,$$

we have at once

$$(P_{n-2}A_{n-1}P_nQ) = (RP_nA_{n+1}P_{n+2}),$$

and the equivalence of the pencils is demonstrated.

Produce $A_{n-1}Q$ to U, making $A_{n-1}U = A_{n-1}O$. Draw $A_{n-1}F$, the bisector of the angle $OA_{n-1}U$; clearly this is parallel to an asymptote. Let OA_n, OA_{n+1} cut $A_{n-1}F$ at H, N respectively. Join HU, NU. Let G, K be the points of intersection of HU with OA_{n-1}, OA_{n+1}, and L, M those of NU with A_{n-1}, A_n respectively. Now HO, HU are equally inclined to $A_{n-1}F$, and similarly so are NO, NU. Therefore NU is parallel to the tangent at A_{n+1}, that is, to RA_{n+1}; also HU is parallel to the tangent at A_n, whence

$$A_{n+1}P_n : P_nR = NM : ML, \quad \text{....................(iv)}$$

and

$$KH = HG. \quad \text{..............................(v)}$$

But since HU and HO are equally inclined to $A_{n-1}F$, therefore

$$HG = HP_n, \quad \text{.........................(vi)}$$

so that HP_n and HK are equal and equally inclined to $A_{n-1}F$; hence P_nK is parallel to $A_{n-1}F$.

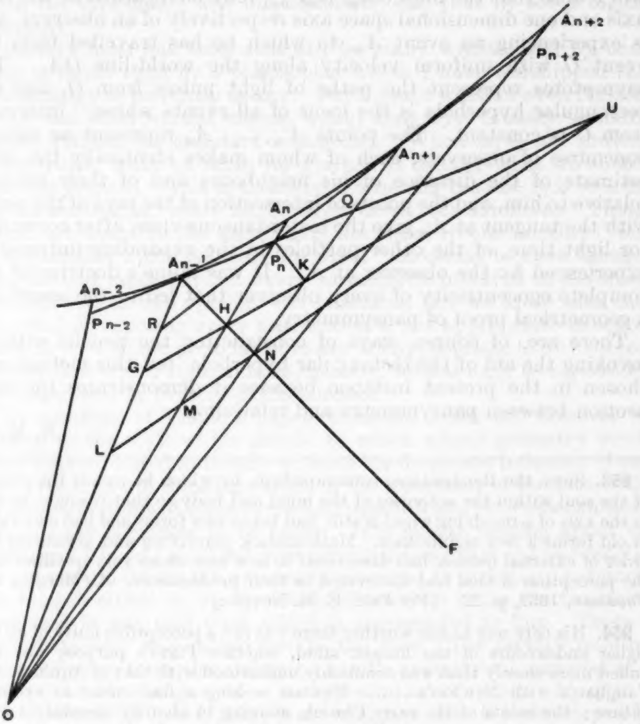

FIG. 2.

It immediately follows that

$$A_{n-1}P_n : P_nQ = NK : KQ$$

$$= NM : ML \text{ (from symmetry about } A_{n-1}F)$$

$$= A_{n+1}P_n : P_nR, \text{ by (iv)},$$

and the properties of pansymmetry are established.

Actually it is immaterial whether the initial point A be the vertex of the hyperbola. The above proof is equally valid if the con-

struction be commenced at any arbitrary point, so that the axis need not be one of the rays of the pencil.

It is of interest to note that the existence of pansymmetrical pencils was first deduced as a natural corollary to Professor E. A. Milne's Extended Principle of Relativity.

Those who are familiar with Minkowski's geometry of space-time will realise that the lines OA_n, A_nP_{n+1} may be regarded as the time axis and one dimensional space axis respectively of an observer, who is experiencing an event A_n, to which he has travelled from the event O with uniform velocity along the world-line OA_n. The asymptotes represent the paths of light pulses from O, and the rectangular hyperbola is the locus of all events whose " interval " from O is constant. The points $A'_n, \ldots, A_n$ represent an infinite concourse of observers, each of whom makes identically the same estimate of the distance of his neighbours and of their velocity relative to him, and the points of intersection of the rays of the pencil with the tangent at A_n give the instantaneous view, after correction for light time, of the other particles in the expanding universe as experienced by the observer at A_n. It was Milne's doctrine of the complete egocentricity of every observer that led to the search for a geometrical proof of pansymmetry.

There are, of course, ways of constructing the pencils without invoking the aid of the rectangular hyperbola, but this method was chosen in the present instance because it demonstrates the connection between pansymmetry and relativity.

B. M. P.

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953. Since the Renaissance, contemplation, by which he meant the stilling of the soul within the activities of the mind and body so that it might be still as the axis of a revolving wheel is still, had taken new forms and had cultivated in old forms a new significance. Mathematics, perceiving and measuring the order of external nature, had discovered in new men those very qualities that the perception of God had discovered in their predecessors.—C. Morgan, *The Fountain*, 1932, p. 22. [Per Prof. E. H. Neville.]

954. His care was to ask whether there was not a perceptible unity of all the higher endeavours of the human mind, whether Plato's purpose was not linked more closely than was commonly understood with that of Aquinas, and Vaughan's with Newton's . . . . Newton seeking a final order in external nature ; the saints of the early Church, striving to identify themselves with God and to lose themselves in Him ; the philosophers who, having no thought of resurrection, devoted their lives to the quest of absolute truth, the untouchable, the timeless thing—were not all these struggling by different paths towards one end, an ecstasy invulnerable because it is " not of the senses " ? —C. Morgan, *The Fountain*, 1932, pp. 26-27. [Per Prof. E. H. Neville.]

955. The philosopher's soul " withdraws itself as far as it can from all association and contact with the body and reaches out after truth by itself ". With what results ? Deprived of its nourishment, the soul grows thin and mangy, like the starved lion. Disgusted and pitying in the midst of our admiration, " Poor brutes ! " we cry at the sight of such extraordinary and lamentable souls as those of Kant, of Newton, of Descartes. " Why aren't they given enough to eat ? "—Aldous Huxley, *Do what you will*, 1929, p. 43. [Per Mr. F. Robbins.]

CONGRUENCE AND PARALLELISM.

EXTRACTS FROM AN ADDRESS TO TEACHERS.*

By E. H. Neville.

One service which Euclid rendered to the school-teacher in the days when his supremacy was unquestioned was, that he set a logical standard.

Euclid's logic has been attacked by many writers, and by none more severely than by a philosopher whom I have worshipped this side idolatry as fervently as any. But I doubt whether even Russell, in his less controversial moods, would deny that broadly speaking Euclid pitches the standard reasonably. When he is not proving the obvious, what he takes for granted is usually what we should take for granted. This is the more interesting because it is to some extent accidental. Since Euclid wrote down such axioms as "Things which are equal to the same thing are equal to one another" and "Magnitudes which coincide are equal" he seems to have thought that he had made explicit every assumption involved in his work. We know that this opinion was mistaken, but we know also that the ideal of exposing every assumption is not appropriate to school geometry—I would go so far as to say not appropriate to any subject except that which it dominates, namely, formal logic as expounded by Peano and in *Principia Mathematica*.

In speaking of Euclid's logical standard I am not however referring to the style of his proofs, to which school geometry would have approximated naturally without the dominant influence of any one writer. I am thinking rather of the treatment of fundamentals, of those questions of congruence and parallels and incommensurables which have somehow to be faced. It is no use beating about the bush, and I hope I shall not give offence if I say that not one teacher in a thousand leaves the university prepared to make any judgment for himself either on the questions themselves or on the treatment proposed in any text-book which is commended to him. There is no reason why this should not be the case. The faculty required is as different from the ability to handle geometrical material as is an aptitude for languages or an appreciation of music. In spite of the comfortable belief, there is no evidence that it is developed by practice in the middle stages of geometry, and nothing is more astounding about Euclid himself than that the man who was capable of proving Pythagoras' theorem by one of the most amazing *tours de force* in the whole of mathematical history was also too acute to make unconsciously any invalid plausible assumptions in building the theory of parallels. One of the reasons which influenced me to join the M.A. Committee on the Teaching of Geometry was that I was interested to hear the teachers with practical experience discussing the form in which questions of this kind are presentable. For although these questions are not even mentioned in an ordinary

* See *Gazette*, Vol. XVII., p. 307.

mathematical course in a university, the teacher of elementary geometry is brought face to face with them very soon, and according to his temperament and his leisure becomes resentful or inquisitive; in either case he needs an acknowledged authority to follow or to learn from. With Euclid and the classical English editors of his work in command there was no difficulty. The methods of dealing with fundamentals were laid down, and one could talk of superposition, quote the parallel axiom or Playfair's, and cut out most of Book I and the whole of Book V, without feeling it necessary to discover why Euclid chose one method rather than another. But when Euclid's authority was repudiated every apparent short cut offered an irresistible temptation, or, to speak more fairly, it became the duty of the author of a text-book to deal with fundamentals as simply and as plausibly as he could, and in the relief from the prolixities and difficulties of Euclid, few waited to ask why Euclid had not taken short cuts which he could not have been so blind as not to see. And since in the nature of things a man who is unaware of difficulties is likely to write a book superficially more readable than a man who is trying to present an easy version of matters which he knows to be really hard, the attractive books are inevitably the dangerous ones. The principle having been admitted that any course that is in print may be followed for examination purposes, the author had everything to gain by an evasive treatment.

Before speaking of individual problems I should like to warn * you against confusing what the teacher should know with what he should endeavour to transmit. When it is said in such work as this that one method avoids a philosophical difficulty which is fatal to another method, there is not the least obligation imposed on the teacher to point out to the pupil the difficulty which is being avoided. It is of the utmost importance that the pupil should have as little as possible to unlearn, and that he should not be bluffed. We must not live in dread of the day when one of our pupils will be acute enough to see through our pretences. But there is no need for us to point out in the class-room how clever we are being: the dull boy will not understand, and we can trust the bright boy to learn sooner or later to appreciate us.

With regard to congruence, understand at once that an assumption of some sort is necessary. Do not, if you can help it, be lured into long discussions in the class-room. Your boys may not object. On the contrary, they may encourage you. The less I am prepared to say to you to-day about geometry, the more ready I shall be to while away time with anecdotes about the committee that drew up the M.A. Report, and the longer they can keep you talking about what you mean by congruence, the longer it will be before they have anything to do themselves. Do not mistake mere lack of intellectual energy for a passion for philosophical subtlety in your boys, however ready you may be to blur the distinction in favour of your

* This warning, sounded by Mr. Goodwill at the Annual Meeting in 1924, was incorporated in the Preface to later editions of the *Report on the Teaching of Geometry*.

grown-up friends at Oxford and elsewhere. But the discussions which should not take place in school and cannot be reported in the school geometry book are none the less discussions in which the teacher should have indulged. They belong, I think, to the training college, and it is one of the functions of such conferences as this to provide opportunities for them to teachers who have not gone through training colleges or have failed at the time to appreciate their importance.

If discussion in the class-room is to be avoided, formal work must begin with a statement either of assumption or of fact. Perhaps it does not matter in which form the existence of congruent figures is put before boys; there is no doubt that it will be accepted as a fact. But it does matter in which form the teacher thinks of it. For if you think of it as a fact, you will be in danger of trying to manufacture the evidence, and you will hardly avoid the question "How can we know that a body is really rigid?" You will be drawn into arguments proper to the physical laboratory, and while this may be less disastrous than philosophy, it is hardly more relevant to your proper business. Think of the existence of congruent figures as an assumption, and you will have no need to establish your right to make it or indeed to define your terms. So long as congruence means something, it does not matter what it means, but the illustrations you accept in actual drawings and models must of course reproduce within the limits of accuracy you attach to them the conditions you assert. I have said that you need not define your terms. There are relations between length, rigidity and congruence, and you can define two of these concepts by means of the third, but one of the three *must* remain undefined. If you fail to realise this, if you attempt to define all three notions, you will certainly be caught into vicious circles, and if you find you cannot escape, you will be tempted at least to hide your entanglement from your class by a fog of words.

The question why your axioms of congruence are plausible, and the nature of their relations to the external world, I might say to the physicist's world, will not trouble your hearers. It is obvious that, applied to concrete diagrams and plasticene models, axioms are not expected to be verified more than to a first approximation—the points and lines and planes are themselves only approximations. It is a simple fact that, to a first approximation, relations are discoverable between some material figures and others which are those postulated in geometry as subsisting between congruent figures; it is further true that we can maintain these relations to a very high degree of accuracy by attributing coefficients of expansion and the like to the substances that occur in the world. It is true nevertheless that observations are observations of matter. The separation between properties of matter and properties of space is our doing, and whether or not it is inevitable philosophically or psychologically that we should make some such separation, the precise division we make is certainly imposed not by any logical necessities but by the desire to find the simplest scheme we can for the expression of natural

phenomena. It is almost a pity that to a first approximation the simplest type of space and the simplest physical laws go together, but now that we are finding that unless we are to complicate the laws beyond endurance we must modify the space, the nature of physical space as a conventional frame that is to some extent subject to our choice is being forced on our attention. Considerations of this kind we shall of course not raise in the elementary class-room, but possibly one of those slack hours between the end of examinations and the end of term might be spent with equal profit and amusement in speculating as to whether the geometry developed by an intelligent jelly-fish with no acquaintance with bodies that we call rigid, swimming in a heterogeneous ocean of variable temperature, would be Euclidean. Would it make any difference if the jelly-fish could see the stars, and is not the difference between the jelly-fish and ourselves quantitative rather than qualitative ?

To resume, congruence for geometry is an indefinable, or, if you prefer it, length is an indefinable. This does not wreck your geometry. On the contrary it leaves it untouched. But it has the advantage of ruling out of order certain questions which would be unanswerable if you consented to listen to them. I say it leaves the geometry untouched, for the form of a mathematical proposition is " If these relations hold, then these other relations hold also " ; your undefined concept must have some properties hypothecated—for formal work on congruence part of Euclid I. 4 is sufficient, but for the second stage of school geometry a great deal more is accepted —and the subsequent deductions take the form " From these properties, others follow ". The questions that are out of order are those which ask how the assumptions in the hypothesis of a theorem are to be verified. We say " If $OP=OQ$, then $OPQ=OQP$ " ; " If C is a right-angle, then $AB^2=AC^2+BC^2$ ". The question " How are we to know beforehand that $OP=OQ$ or that C is a right-angle ? " is improper. At best, taken as referring to an actual figure, it implies an acceptable standard of accuracy and becomes " How are we to test the delicacy of our apparatus and the rigidity of our materials ? " and is clearly a question for the physicist. But taken as referring to pure geometry, the question is irrelevant : there is no need to go behind the assumption you are making.

The manner in which the idea of congruence is utilised and brought into practice is explained sufficiently in the Report and I need not dwell on it. Little is wanted but a modification of the language Euclid uses of superposition.

Of the theory of parallels I should have preferred to say little, but the subject illustrates with devastating clearness the danger of setting the average geometer free to disseminate fallacies. Euclid's theory was based on a postulate that nobody has ever defended as plausible, but the theory is logically unimpeachable and there is no temptation to introduce fallacies anywhere, once the general outline is accepted. The attempts to find a plausible substitute for the postulate are not wholly a service to Euclid's system. If an assump-

tion is, as we now know Euclid's to be, in fact arbitrary, from the point of view of pure geometry, it is better adopted arbitrarily than made to seem inevitable.

I venture to think that for most of us the difficulty comes from a failure to distinguish between two questions, which, not unnaturally, have different answers.

The first question is, in the realm of pure geometry, are the properties of Euclidean parallels inseparable from the properties postulated of congruent figures ? The second is, if the external world is reduced by abstraction to bodies departing from rigidity according to definite laws, so that congruence is assertable of actual figures, is the physical space which is then made amenable to measurement a space in which the parallel axiom holds ?

Euclid himself must have suspected that the question in pure geometry must be answered in the negative ; at least the answer is known definitely now. The axioms of congruence do not imply the parallel postulate. This is as certain a theorem of pure mathematics as, let us say, that the product $x\,(1-x^2/\pi^2)\,(1-x^2/2^2\pi^2)\ldots$ and the series $x-x^3/3\,!+x^5/5\,!-\ldots$ tend to the same limit.

The second question admits of very little more doubt. The assumption that physical bodies obey simple laws in a Euclidean space does work, to so high a degree of approximation that it is the assumption we cannot help trying first. In physical space, as we apply to it the vocabulary of pure geometry, no departure from Euclidean behaviour is within the powers of unaided observation to detect. But observation, aided or unaided, has its limits, and can give no proof of a theoretical result transcending those limits. We may be satisfied that the sum of the angles of a particular triangle whose sides are light rays does not differ from two right-angles by more than one ten-thousandth of a second ; that does not disprove a difference of one-millionth, and when millionths are in our grasp, thousand-millionths will remain undetected : the chase of the theoretical by the practical reproduces the famous chase of the tortoise by Achilles, but with an insuperable difficulty that is no paradox.

The fact that within the range of observation Euclidean geometry is consonant with simple physical laws, explains why no deductions can be made from the plausibility of the strange axiom by which Dodgson in his *New Theory of Parallels* proposed to replace Euclid's. In the first edition of the book, the axiom is that for some definite value of n, $2^n\times$the area of a regular hexagon is greater than the area of the segment of the circumscribing circle cut off by one side of the hexagon ; in the third edition the hexagon is replaced by a square, or rather, since to assume the existence of a square as defined by Euclid is already to assume that the plane is Euclidean, by a regular tetragon—a four-sided figure whose sides are equal and whose angles also are equal but are not assumed to be right-angles.

Dodgson was above all else a logician, and his logic is not faulty ; his axiom, with the axiom that the straight line is of infinite length,

which he never thought of questioning, does imply Euclidean geometry, and therefore, conversely, in non-Euclidean geometry one or other of these axioms does not hold. In the elliptic plane, Dodgson's axiom is satisfied, but his arguments break down because the line is re-entrant. In the hyperbolic plane, the angles of the tetragon are acute, and the larger the figure, the smaller these angles. The figure is rather like the figure obtained by moving each branch of a rectangular hyperbola and each branch of the complementary hyperbola towards the common centre and suppressing the arcs beyond the points where adjacent branches intersect. If the shift inwards is so large that the distance of the vertices of the branches from the centre becomes small compared with the semiaxis of the curves, the figure approximates to a square; but if the shift is small compared with the semiaxis, the distance of the corners from the centre is large compared with the distance of the vertices from the centre, and the area inside the star is small compared with the area of the circumcircle. This was pointed out to Dodgson, who replied that the phenomenon did not appal him so much as might have been expected: naturally an axiom which implies Euclidean geometry is not satisfied in any other geometry, if there is any other geometry. This is outrageous bluff. The question is not logical but psychological. Why is the axiom plausible—so plausible that to Dodgson it seemed to disprove the possibility of a geometry in which it is contradicted?

I suggest that the only reason why Dodgson's axiom is more plausible than Euclid's and than other substitutes for Euclid's is that whereas these refer explicitly to what can or cannot happen at distances indefinitely great, Dodgson's axiom seems to refer to a single closed figure. Had he proposed to assume that the ratio of the area of the tetragon to the area of an excluded segment of the circumcircle, though possibly diminishing as the figure increases in size, has some lower limit which is not zero, the whole content of the axiom would not have seemed to be contained in a single diagram, and he could not have maintained for a moment that his assumption was more plausible than Euclid's. With tears in his eyes and a sob in his throat he asks to be introduced to the gentle reader who thinks it just possible to squeeze 512 tetragons into the segment but is willing to allow that no amount of skilful packing will dispose of 1024 of them. The fallacy is latent if not patent in the form of the appeal: an inequality which is granted for a particular tetragon under observation is declared to hold for every tetragon. Allowing the assumption to take tacitly the form that the ratio of one area to the other had some *constant* value, to say that the constant is not zero is merely to admit that the tetragon is not identical with its circumcircle; think of a variable ratio, recognise that Dodgson ought to rival Euclid's rhetoric by speaking of what may happen in a tetragon *however great*, and there is no plausibility in his proposal, no force in his appeal.

The assumption that the ratio is constant slips through unchallenged, because it is essentially the assumption that shapes and

relative sizes are preservable, that is, that similar figures exist*. This is an assumption we are all ready to make, for the adequate reason that we are acting on it throughout our daily lives, but the fact that we find it satisfactory locally, or in other words that we are small in the world, does not imply that geometry without it is logically impossible, while the fact that it is equivalent to the axiom of parallels was pointed out by Wallis long ago.

There is no fear that axioms as bizarre as Dodgson's will ever be adopted seriously, and the teacher who has understood that an element of postulation is necessary even in regard to the existence of congruent figures is not likely to give a higher status to the existence of similar figures. The really dangerous theory of parallels is the direction theory, exposed brilliantly and wittily by Dodgson himself, refuted again and again, but continually revived by the "practical" teacher; in this theory the fundamental assumption is always hidden instead of being displayed. The fallacies are so various in form that I cannot attempt to deal with them, and I will conclude by saying quite seriously that any teacher who has followed a few such arguments on this subject as are in the M.A. Report, and is nevertheless unable to discover for himself the fallacy in any theory of parallels which claims to dispense with an assumption equivalent to Euclid's, had better leave to a colleague the difficult beginnings of geometry; he will be a better and a more inspiring teacher of mathematics if he gives up the attempt to exercise a faculty which he does not possess, and which is, as I said an hour ago, utterly distinct from the power to handle geometrical material.

E. H. N.

CORRESPONDENCE.

Dec. 13th, 1933.

To the Editor of the *Mathematical Gazette*.

Dear Sir,—With regard to the erroneous article in my *Coordinate Geometry*, noticed by "A. R." in his recent review in your columns, I should be greatly obliged if you would let your readers know that the error was found almost immediately on publication, and that a correct version was printed off on two pages, 95, 96, of which I have several copies. The erroneous article only appears in the review copies and the first fifty or so that were sold.

If anyone possesses one of these copies, in which an *inequality* forms the last line of page 95 instead of the equations,

$$x'/a' = y'/b' = -c'/(a'^2 + b'^2),$$

I shall be only too pleased to send them a copy of this pair of pages, so that they can cut out the erroneous page, leaving a tab formed by the inner white margin, to which the new sheet can be pasted.

Yours very faithfully,

J. M. Child.

* This too was said to Dodgson, who missed the point childishly, and protested that as he had nowhere equated the angles of different tetragons, the question of whether tetragons of different sizes are similar was irrelevant. But his figure consists not of a tetragon alone, but of tetragon and circle. The question is: Why is his axiom plausible? And the answer: Because it is true for figures sufficiently small.

THE OPERATORS E AND Δ.

BY S. BARNARD.

1. REFERRING to Mr. Freeman's article in the July number of the *Gazette*, there can be no doubt as to the desirability of introducing these operators early in algebra. But they should be used only for the transformation of *finite* series. For this purpose it is *never* necessary to use negative powers of E and Δ. Some of Mr. Freeman's examples are considered below from this point of view.

So far as addition, subtraction and multiplication are concerned, E and Δ, operating on a *single definite function*, combine with themselves, with each other and with constants according to the laws of algebra. It is a totally different thing to say, as Mr. Freeman does, that

$$\Delta(u_x + v_x + w_x + \dots) = \Delta u_x + \Delta v_x + \Delta w_x + \dots.$$

In fact, this statement is true only when the number of terms is independent of x. For example, if

$$s_n = u_1 + u_2 + \dots + u_n,$$

then
$$\Delta s_n = s_{n+1} - s_n = u_{n+1},$$

but
$$\Delta u_1 + \Delta u_2 + \dots + \Delta u_n = u_2 - u_1 + u_3 - u_2 + \dots + u_{n+1} - u_n$$
$$= u_{n+1} - u_1.$$

2. The question of using a function of E or Δ as a divisor requires some consideration. As an example, take the transformation

$$u_1 + u_2 + \dots + u_n = nu_1 + \frac{n(n-1)}{2!}\Delta u_1 + \dots + \Delta^{n-1} u_1. \quad \dots\dots\dots\text{(i)}$$

Mr. Freeman quotes the usual proof, which begins as follows: if $s_n = u_1 + u_2 + \dots + u_n$, we have

$$s_n = (1 + E + E^2 + \dots + E^{n-1})\, u_1$$
$$= \frac{E^n - 1}{E - 1} u_1, \text{ etc.}$$

The method gives the right result, but what about the intermediate steps? The student may argue thus: if the equation

$$s_n = \frac{E^n - 1}{E - 1} u_1$$

has any meaning at all, it must mean that

$$(E-1)\, s_n = (E^n - 1)\, u_1.$$

But $(E-1)\, s_n = u_{n+1}$, and $(E^n - 1)\, u_1 = u_{n+1} - u_1$, and so this step seems to be wrong.

It should be pointed out that the operator is written in a fractional form in order to transform a polynomial in E into a polynomial in Δ. The result, when obtained, can be verified by multiplication and addition only, which justifies the process: but at every stage *the operator must be regarded as a whole.*

In such transformations the operator may be put in the form $f(\Delta)/\phi(\Delta)$, where $f(\Delta)$ and $\phi(\Delta)$ are polynomials in Δ, *provided that* $f(\Delta)$ is divisible by $\phi(\Delta)$ and $f(\Delta)$, $\phi(\Delta)$ are not regarded as separate operators.

The transformation (i) is proved most simply as follows.

We have $u_n = E^{n-1}u_1 = (1+\Delta)^{n-1}u_1$, therefore

$$u_n = u_1 + (n-1)\Delta u_1 + \tfrac{1}{2}(n-1)(n-2)\Delta^2 u_1 + \dots + \Delta^{n-1}u_1.$$

Similarly

$$u_{n-1} = u_1 + (n-2)\Delta u_1 + \tfrac{1}{2}(n-2)(n-3)\Delta^2 u_1 + \dots + \Delta^{n-2}u_1.$$

$$\dots\dots\dots$$

$$u_2 = u_1 + \Delta u_1.$$

$$u_1 = u_1.$$

Whence by addition

$$s_n = nu_1 + \tfrac{1}{2}n(n-1)\Delta u_1 + \dots + \Delta^{n-1}u_1.$$

Another proof, in which division is avoided, is given in Chrystal's *Algebra*, ii, p. 405. It may be noticed that, although Chrystal introduces E and Δ rather late in his book, he limits their application to finite series and avoids the use of negative powers of these symbols.

3. I now consider some of Mr. Freeman's examples, with alternative proofs, avoiding the use of E^{-r}, Δ^{-r}.

Ex. 1. If $(1+x)^n = c_0 + c_1x + \dots + c_nx^n$, where n is a positive integer, find the value of

$$(n-1)^2c_1 + (n-3)^2c_3 + (n-5)^2c_5 + \dots .$$

If $u_n = n^2$ and S is the sum of the series,

$$\begin{aligned} S &= (c_1E^{n-1} + c_3E^{n-3} + \dots)\, u_0 \\ &= \tfrac{1}{2}\{(E+1)^n - (E-1)^n\}\, u_0, \text{ for } c_0 = c_n,\ c_1 = c_{n-1}, \text{ etc.,} \\ &= \tfrac{1}{2}\{(\Delta+2)^n - \Delta^n\}\, u_0 \\ &= \tfrac{1}{2}\{2^n u_0 + n\,.\,2^{n-1}\Delta u_0 + \tfrac{1}{2}n(n-1)\,.\,2^{n-2}\Delta^2 u_0\} \end{aligned}$$

for $\Delta^r u_0 = 0$ if $r > 2$; now $u_0 = 0$, $\Delta u_0 = 1$, $\Delta^2 u_0 = 2$, thus

$$S = 2^{n-3}n(n+1).$$

Ex. 3 (extended). Show that, if $c_0, c_1, \dots$ are as in the last example,

$$c_0x^n - c_1(x-y)^n + c_2(x-2y)^n \dots + (-)^n c_n(x-ny)^n = y^n\,.\,n\,!.$$

Let $u_r = (x-ry)^n$ and let S be the sum of the series, then if E, Δ apply to r,

$$\begin{aligned} S &= (c_0 - c_1E + c_2E^2 - \dots)\, u_0 \\ &= (1-E)^n u_0 \\ &= (-)^n\Delta^n u_0. \end{aligned}$$

Now u_r is a polynomial in r of which the highest term is $(-)^n r^n y^n$. Hence $\Delta^n u_r = (-)^n y^n\,.\,n\,!$ and $S = y^n\,.\,n\,!$

Ex. 4. Sum the series

$$\frac{1}{a+1}+\frac{2\,!}{(a+1)(a+2)}+\ldots+\frac{n\,!}{(a+1)(a+2)\ldots(a+n)}.$$

Here the rth term can at once be put in the form $v_r - v_{r-1}$, and it is not worth while to introduce E and Δ.

In § 6 the introduction of E_1, E_2 is quite unnecessary, and Ex. 6 may be done as follows.

To find the value of

$$S \equiv xy-(r+1)(x-1)(y-1)+\frac{r(r+1)}{2\,!}(x-2)(y-2)-\ldots \text{ to } r+2 \text{ terms.}$$

Let $u_n=(x-n)(y-n)$, then if E, Δ apply to n,

$$S=(1-E)^{r+1}u_0=(-)^{r+1}\Delta^{r+1}u_0.$$

Since u_n is a quadratic function of n, $\Delta^{r+1}u_n=0\,(r>1)$, and $\Delta^2 u_0=2$, $\Delta u_0=-x-y+1$: therefore $S=x+y-1$, 2 or 0 according as $r=0$, $r=1$ or $r>1$. Mr. Freeman does not give the complete answer.

Ex. 10. The proof given is difficult, its validity is doubtful, and obviously the sum of the series is twice the coefficient of x^{n-1} in $(1-x)^{-3}\,.\,(1-x)^{-n}$.

4. In connection with series, the following result is often useful and is not given in the text-books.

If $u_0=1$, $u_1=\dfrac{a}{b}$, $u_2=\dfrac{a(a-1)}{b(b-1)}$, ..., $u_n=\dfrac{a(a-1)\ldots(a-n+1)}{b(b-1)\ldots(b-n+1)}$,

then, if $n \geqslant 1$,

$$\Delta^r u_n=\frac{a-b}{b}\cdot\frac{a-b+1}{b-1}\ldots\frac{a-b+r-1}{b-r+1}$$
$$\times\frac{a(a-1)\ldots(a-n+1)}{(b-r)(b-r-1)\ldots(b-r-n+1)},$$

and
$$\Delta^r u_0=\frac{a-b}{b}\cdot\frac{a-b+1}{b-1}\ldots\frac{a-b+r-1}{b-r+1}.$$

Ex. Prove that

$$\left(1-\frac{a}{b}\right)\left(1-\frac{a}{b-1}\right)\left(1-\frac{a}{b-2}\right)\ldots\left(1-\frac{a}{b-n+1}\right)$$
$$=1-n\cdot\frac{a}{b}+\frac{n(n-1)}{2\,!}\cdot\frac{a(a-1)}{b(b-1)}-\ldots \text{ to } n+1 \text{ terms.}$$

5. Later on, *as a means of discovery*, these processes may be applied to infinite series *in a purely formal way* and without any reference to convergence. Any result so obtained must be tested by the use of double series, or otherwise, before it can be regarded as true.

Thus the result in Ex. 8 of Mr. Freeman's article can be verified for all values of x by the multiplication of series.

Or again, Montmort's transformation of the series $\sum_1^\infty u_n x^n$ may be found by the formal reckoning below.

$$u_1x+u_2x^2+\ldots+u_nx^n+\ldots$$
$$=x(1+Ex+E^2x^2+\ldots)u_1$$
$$=\frac{x}{1-Ex}u_1$$
$$=\frac{y}{1+y-Ey}u_1, \text{ where } x=y/(1+y)$$
$$=\frac{y}{1-\Delta y}u_1$$
$$=u_1y+\Delta u_1 \cdot y^2+\Delta^2 u_1 \cdot y^3+\ldots,$$

and since $y=x/(1-x)$, we have

$$\sum_1^\infty u_nx^n=u_1\cdot\frac{x}{1-x}+\Delta u_1\cdot\frac{x^2}{(1-x)^2}+\Delta^2u_1\cdot\frac{x^3}{(1-x)^3}+\ldots.$$

If $\Sigma u_n x^n$ is convergent for $|x|<1$, by using double series, we can show that the transformation holds if $-1<x<\frac{1}{2}$. By Abel's theorem it also holds when $x=-1$, provided that $\Sigma(-)^n u_n$ is convergent.

6. Another interesting example is the following theorem, due to Cayley (*Coll. Math. Papers*, ix, 259-263).

If $$\frac{t}{e^t-1}=1-B_1t+B_2\frac{t^2}{2!}-B_3\frac{t^3}{3!}+\ldots$$

so that $(-)^{n-1}B_{2n}$ is the nth number of Bernoulli, then

$$(-)^rB_r=(1-\tfrac{1}{2}\Delta+\tfrac{1}{3}\Delta^2-\ldots)0^r.$$

Cayley's proof. We have

$$e^{t.0}=1+t.\frac{0}{1!}+t^2.\frac{0^2}{2!}+\ldots$$

and $$e^t-1=e^{t.1}-e^{t.0}=\Delta e^{t.0},$$

therefore $$e^t=(1+\Delta)e^{t.0}$$

and $$t=\log(1+\Delta)e^{t.0}.$$

Hence $$\frac{t}{e^t-1}=\frac{\log(1+\Delta)}{\Delta}e^{t.0}$$

and $$1-B_1.t+B_2.\frac{t^2}{2!}-\ldots$$
$$=(1-\tfrac{1}{2}\Delta+\tfrac{1}{3}\Delta^2-\ldots)\left(1+t.\frac{0}{1!}+t^2.\frac{0^2}{2!}+\ldots\right).$$

The theorem follows on equating the coefficients of t^r. This proof requires some justification.

A simpler proof. Let

$$S_r = 1^r + 2^r + 3^r + \dots + n^r,$$

then S_r can be expressed as a polynomial in n *in which* B_r *is the coefficient of* n.* (This follows from Bernoulli's theorem, which expresses S_r in terms of $B_1, B_2, B_3, \dots$.) Also

$$n^r = (1+\Delta)^n 0^r$$

$$= \left(n\Delta + \frac{n(n-1)}{2!}\Delta^2 + \dots + \Delta^n\right)0^r, \quad \text{(ii)}$$

therefore

$$S_r = \left\{\frac{(n+1)n}{2!}\Delta + \frac{(n+1)n(n-1)}{3!}\Delta^2 + \dots + \Delta^n\right\}0^r.$$

Hence

$$B_r = \left\{\frac{1}{2!}\Delta - \frac{1!}{3!}\Delta^2 + \frac{2!}{4!}\Delta^3 - \dots + (-)^{r-1}\frac{(r-1)!}{(r+1)!}\Delta^r\right\}0^r$$

$$= \left\{(1-\tfrac{1}{2})\Delta - (\tfrac{1}{2}-\tfrac{1}{3})\Delta^2 + \dots + (-)^{r-1}\left(\frac{1}{r} - \frac{1}{r+1}\right)\Delta^r\right\}0^r. \quad \text{(iii)}$$

Again, by equating the coefficients of n in (ii), we have, if $r > 1$,

$$\left(\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \dots + (-)^{r-1}\frac{\Delta^r}{r}\right)0^r = 0.$$

Hence by (iii)

$$B_r = \left(1 - \frac{\Delta}{2} + \frac{\Delta^2}{3} - \dots + (-)^{r-1}\frac{\Delta^r}{r+1}\right)0^r, \quad (r > 1);$$

and since $B_1 = \frac{1}{2}$, and $B_3, B_5, \dots$ are all zero,

$$(-)^r B_r = (1 - \tfrac{1}{2}\Delta + \tfrac{1}{3}\Delta^2 - \dots)0^r \quad \text{for } r > 0.$$

Remark. Equation (iii) can be written

$$B_r = (1 - \tfrac{1}{2}\Delta + \tfrac{1}{3}\Delta^2 - \dots)(1+\Delta)0^r.$$

Hence it is also true that

$$B_r = (1 - \tfrac{1}{2}\Delta + \tfrac{1}{3}\Delta^2 - \dots)1^r.$$

For example, using the table of differences

$$\begin{array}{c} 0,\ 1,\ 4,\ 9\dots \\ 1,\ 3,\ 5\dots \\ 2,\ 2\dots \end{array}$$

we obtain

$$B_2 = (1 - \tfrac{1}{2}\Delta + \tfrac{1}{3}\Delta^2)0^2$$
$$= -\tfrac{1}{2} + \tfrac{2}{3} = \tfrac{1}{6}.$$

Also

$$B_2 = (1 - \tfrac{1}{2}\Delta + \tfrac{1}{3}\Delta^2)1^2$$
$$= 1 - \tfrac{3}{2} + \tfrac{2}{3} = \tfrac{1}{6}.$$

S. BARNARD.

* I prefer to start with this as the *definition* of B_r. Using only elementary methods, we can then deduce very shortly a large number of properties of the Bernoullian numbers.

ON RECURRING CONTINUED FRACTIONS.

By H. Orfeur.

1. *Notation.* By $(x/y,\ X/Y)=a_1{\cdot}a_2a_3\ldots a_n$, let it be understood that after a vulgar fraction X/Y has been converted into a continued fraction (C.F.), $a_1{\cdot}a_2, a_3 \ldots$ are the partial quotients, and x/y is the penultimate convergent. The point placed after a_1 is intended to show that, in the case of a proper fraction, the *first* partial quotient must be taken as zero.

2. *Extension Rule.* If C.F. $(a/b,\ A/B)$ is to be extended by another $(x/y,\ X/Y)$, the resulting C.F. will be

$$\left(\frac{ax+Ay}{bx+By},\ \frac{aX+AY}{bX+BY}\right).$$

Call the first fraction "the base" and the second "the complement".

In the result the first partial quotient of the complement will be added to the last one of the base.

Example. If $5{\cdot}243=(49/9,\ 158/29)$ is to be extended by

$$6{\cdot}18=(7/1,\ 62/9),$$

the result will be

$$(501/92,\ 4460/819)=5{\cdot}24918.$$

3. (1) The condition that $(a/b,\ A/B)$ may be a complement of $(u/v,\ U/V)$ is that the four determinants,

$$g=\begin{vmatrix} u & b \\ U & B \end{vmatrix},\quad h=\begin{vmatrix} v & b \\ V & B \end{vmatrix},\quad G=\begin{vmatrix} a & u \\ A & U \end{vmatrix},\quad H=\begin{vmatrix} a & v \\ A & V \end{vmatrix},$$

must have the same sign.

This may be verified by using the extension rule upon the base $(kg/kh,\ kG/kH)$, in which $k=aB-bA=\pm 1$. In the result the last convergent will be k^2U/k^2V.

The reason for the factor k is that it renders each numerator and each denominator positive.

(2) The condition that $(a/b,\ A/B)$ may be a base of $(u/v,\ U/V)$ is that the four determinants

$$m=\begin{vmatrix} u & A \\ v & B \end{vmatrix},\quad n=\begin{vmatrix} a & u \\ b & v \end{vmatrix},\quad M=\begin{vmatrix} U & A \\ V & B \end{vmatrix},\quad N=\begin{vmatrix} a & U \\ b & V \end{vmatrix},$$

must have the same sign.

For with the supposed base and the extension $(km/kn,\ kM/kN)$ the rule will give the same result.

4. Scheme (1) $k\,.\,\dfrac{\begin{vmatrix} u & b \\ U & B \end{vmatrix}\begin{vmatrix} a & u \\ A & U \end{vmatrix}}{\begin{vmatrix} v & b \\ V & B \end{vmatrix}\begin{vmatrix} a & v \\ A & V \end{vmatrix}}$; Scheme (2) $k\,.\,\dfrac{\begin{vmatrix} u & A \\ v & B \end{vmatrix}\begin{vmatrix} U & A \\ V & B \end{vmatrix}}{\begin{vmatrix} a & u \\ b & v \end{vmatrix}\begin{vmatrix} a & U \\ b & V \end{vmatrix}}$.

Conversely, if it is known that A/B is a complement of U/V, then the corresponding base is given by Scheme (1).

Or, if it is known that A/B is a base of U/V, then the corresponding complement is given by Scheme (2).

5. Suppose A/B, A'/B' are two fractions, the former being the greater. If they are converted into C.F.'s, a new C.F. (<1) may be obtained according to the scheme

$$\frac{\left|\begin{array}{cc} a & a' \\ b & b' \end{array}\right| \; \left|\begin{array}{cc} A & a' \\ B & b' \end{array}\right|}{\left|\begin{array}{cc} a & A' \\ b & B' \end{array}\right| \; \left|\begin{array}{cc} A & A' \\ B & B' \end{array}\right|} = \left(\frac{e}{f},\ \frac{E}{F}\right), \text{ say. } \qquad \text{......................(3)}$$

We shall call this the "derived continued fraction".

6. When a fraction involving a quadratic surd is converted into a C.F. by the well-known rule, the result has a period (cycle) of recurring partial quotients, usually preceded by some non-recurring ones (acyclic part).

Let $x = a_1 \cdot a_2 a_3 \ldots a_r \dot{b}_1 b_2 \ldots \dot{b}_n$ represent such an expression.

Points are placed over the beginning and the end of the period.

Suppose that the acyclic part $= (a/b,\ A/B)$, ...(4)

$$a_1 \cdot a_2 \ldots b_n = (u/v,\ U/V), \qquad \text{......................(5)}$$

$$0 \cdot b_1 b_2 \ldots b_n = (p/q,\ P/Q), \qquad \text{......................(6)}$$

$$y = 0 \cdot \dot{b}_1 b_2 \ldots \dot{b}_n = \frac{py+P}{qy+Q},$$

then
$$qy^2 + (Q-p)y - P = 0. \qquad \text{..............................(7)}$$

7. Let $\qquad (a_r - b_n)/(a_r \sim b_n) = j = \pm 1.$

Now
$$g/bu = a_r - b_n \pm \text{ a proper fraction.}$$
$$h/bv = a_r - b_n \pm \quad \text{,,} \quad \text{,,}$$
$$G/au = b_n - a_r \pm \quad \text{,,} \quad \text{,,}$$
$$H/av = b_n - a_r \pm \quad \text{,,} \quad \text{,,}$$

Hence in general g, h will have the same sign, and G, H will have the sign opposite to that of g, h.

8. $x = \dfrac{ay+A}{by+B} = \dfrac{uy+U}{vy+V}$. The eliminant of y from these equations is

$$hx^2 + (H-g)x - G = 0. \qquad \text{........................(8)}$$

The surd involved in the roots is $\sqrt{(H+g)^2 - 4(gH - hG)}$.

The extension rule gives $u = ap + Aq, \quad U = aP + AQ,$

$$v = bp + Bq, \quad V = bP + BQ.$$

If these expressions are substituted for u, v, U, V, in g, h, G, H, equation (8) becomes

$$(b^2P - B^2q + bBR)x^2 - (2abP - 2ABq + cR)x$$
$$+ (a^2P - A^2q + aAR) = 0, \ \ldots(9)$$

where $R = Q - p$, $c = aB + bA$.

9. If the quadratic equation has both roots numerically less than 1, transform it into its reciprocal equation.

CLASS I. Roots having opposite signs.

Here h and G must have the same sign; thus condition (1) of § 3 is satisfied. Consequently the partial quotients of the supposed acyclical must form a complement of (U/V) and of (P/Q). Hence either root can be expressed as a recurring C.F. without an acyclical. Thus equation (8) gets back to (7) without the tacit limitation that $P < Q$.

From this may be deduced that the root, when converted into a C.F. in the only possible way, shows an acyclical having a single quotient which is *less* than the last quotient of the period.

Thus if $$x = a_1 \cdot \dot{b}_2 b_3 \dots \dot{b}_n, \quad \text{(i)}$$
it may be expressed as
$$x = \dot{a}_1 \cdot b_2 b_3 \dots \dot{c}. \quad \text{(ii)}$$
where $c = b_n - a_1$. *E.g.*, $3\cdot\dot{4}26\dot{8} = \dot{3}\cdot426\dot{5}$.

10. Equation (7) shows that we may interchange Q and p without altering the numerical roots. But this interchange reverses the order of the quotients; hence that reversal produces the other root.
Taking (ii) as one root, the other must be
$$-\dot{c}\cdot b_{n-1} \dots b_2 \dot{a}_1 = -c\cdot\dot{b}_{n-1} \dots b_2 \dot{b}_n.$$
It follows, therefore, that the last quotient of the period when expressed in the usual way (i) = the *sum of the integers of the roots.*

Examples. (1) If
$$x = -2\cdot\dot{4}16\dot{5} = -\dot{2}\cdot416\dot{3},$$
$$\text{the other root} = \dot{3}\cdot614\dot{2} = 3\cdot\dot{6}14\dot{5}.$$
These are the roots of $17x^2 - 16x - 118 = 0$.

(2) If
$$x = -0\cdot\dot{4}36\dot{2},$$
$$\text{the other root} = \dot{2}\cdot634\dot{0} = 2\cdot\dot{6}34\dot{2}.$$
These are the roots of $82x^2 - 158x - 41 = 0$.

The zero partial quotient at the end of the period points to the fact that in this case P and Q are less than p and q, respectively.

11. Lastly there is the interesting case in which the roots are numerically equal. Here the middle term of (7) is absent; so $Q = p$; and hence the period is symmetrical, or, as Prof. Chrystal has it, " is reciprocal ".
Consequently, the root, in its usual form (i), must have the last quotient of the period *double the integer of the root.*

Example. Numerically, each root of $8x^2 - 159 = 0$, is
$$\dot{4}\cdot252\dot{4} = 4\cdot\dot{2}52\dot{8}.$$

12. Class II. **Roots having the same sign.**

Here h and G must not have the same sign; therefore the partial quotients of the acyclical cannot appear as a complement of the period.

Let x' be the second root, and all symbols used above, now distinguished by dashes, have the same significations with respect to x'.

By $j\,(g/h,\ -G/-H)$, let it be understood that each numerator and each denominator should be multiplied by j (which stands for ± 1), so as to make them positive and produce two true convergents.

It is assumed in § 6 that A/B is a convergent of U/V; hence

$$k\,(m/n,\ M/N) = (p/q,\ P/Q).$$

13. $$(a/b,\ A/B) \text{ is a base of } j\,(g/h,\ -G/-H). \qquad \text{.........(10)}$$

This may be proved as follows:

The complement would be

$$kj\,.\,\frac{\begin{vmatrix} g & A \\ h & B \end{vmatrix} \begin{vmatrix} A & G \\ B & H \end{vmatrix}}{\begin{vmatrix} a & g \\ b & h \end{vmatrix} \begin{vmatrix} G & a \\ H & b \end{vmatrix}} = kj\,.\,\frac{\begin{vmatrix} m & b \\ M & B \end{vmatrix} \begin{vmatrix} m & a \\ M & A \end{vmatrix}}{\begin{vmatrix} n & b \\ N & B \end{vmatrix} \begin{vmatrix} n & a \\ N & A \end{vmatrix}} = j\,.\,\frac{\begin{vmatrix} p & b \\ P & B \end{vmatrix} \begin{vmatrix} p & a \\ P & A \end{vmatrix}}{\begin{vmatrix} q & b \\ Q & B \end{vmatrix} \begin{vmatrix} q & a \\ Q & A \end{vmatrix}}. \qquad (11)$$

. These last four determinants have the same sign, because if each be divided by the product of their small symbols, the four results will be found equal in respect to their integral terms.

Because the above acyclical is a base of (10), it follows that the second acyclical is a base of (10) reversed; therefore $(b'/B',\ a'/A')$ is a complement of (10).

On removing this complement from (11), there results:

$$-k'j\,.\,\frac{\begin{vmatrix} Bp-bP, & B' \\ Ap-aP, & A' \end{vmatrix} \begin{vmatrix} b', & Bp-bP \\ a', & Ap-aP \end{vmatrix}}{\begin{vmatrix} Bq-bQ, & B' \\ Aq-aQ, & A' \end{vmatrix} \begin{vmatrix} b', & Bq-bQ \\ a', & Aq-aQ \end{vmatrix}} = k'j\,.\,\frac{\begin{vmatrix} p & f \\ P & F \end{vmatrix} \begin{vmatrix} e & p \\ E & P \end{vmatrix}}{\begin{vmatrix} q & f \\ Q & F \end{vmatrix} \begin{vmatrix} e & q \\ E & Q \end{vmatrix}}, \qquad (12)$$

in which e, f, E, F have the meanings given them in § 5.

On comparing (12) with Scheme (1), we remark that the continued fraction derived from the acyclicals is a complement of the period.

The corresponding base is (12).

We call these the "reversible base" and the "reversible complement". To exhibit the above results simply we will suppose that the period has five quotients (their number does not matter) $b_1 b_2 b_3 b_4 b_5$; the last two forming the reversible complement and that the acyclicals have three quotients each, viz. $a_1{\cdot}a_2 a_3$ and $a_1'{\cdot}a_2' a_3'$.

Then $j\,(g/h,\ -G/-H) = a_1{\cdot}a_2 a_3 b_1 b_2 b_3 a_3' a_2' a_1'$. ....(10)

C.F. $(12) = 0{\cdot}b_1 b_2 b_3$.

$$\text{The reversal of } (11) = j\,.\,\frac{\begin{vmatrix} p & a \\ P & A \end{vmatrix} \begin{vmatrix} q & a \\ Q & A \end{vmatrix}}{\begin{vmatrix} p & b \\ P & B \end{vmatrix} \begin{vmatrix} q & b \\ Q & B \end{vmatrix}} = a_1'{\cdot}a_2' a_3' b_3 b_2 b_1. \qquad \text{...(13)}$$

$$\left.\begin{aligned} x &= a_1 \cdot a_2 a_3 \dot{b}_1 b_2 b_3 b_4 \dot{b}_5 \\ x' &= a_1' \cdot a_2' a_3' \dot{b}_3 b_2 b_1 b_5 \dot{b}_4 \end{aligned}\right\}.$$

Example. Given $x = 2\cdot13\dot{2}17534\dot{4}$, find x'.

Here acyclical $=(3/1,\ 11/4)$ and period $=(131/377,\ 565/1626)$.

$$\text{C.F. } (13) = \frac{\begin{vmatrix} 131, & 3 \\ 565, & 11 \end{vmatrix} \quad \begin{vmatrix} 377, & 3 \\ 1626, & 11 \end{vmatrix}}{\begin{vmatrix} 131, & 1 \\ 565, & 4 \end{vmatrix} \quad \begin{vmatrix} 377, & 1 \\ 1626, & 4 \end{vmatrix}} = (254/41,\ 731/118) = 6\cdot5712.$$

All except the first of these quotients, reversed, belong to the given period; thus the reversible complement is $\cdot34$;

hence $$x' = 6\cdot\dot{5}7124\dot{3}.$$

To check: Get derived C.F. from acyclicals, thus:

$$\frac{\begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} \quad \begin{vmatrix} 6 & 3 \\ 1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 11 \\ 0 & 4 \end{vmatrix} \quad \begin{vmatrix} 6 & 11 \\ 1 & 4 \end{vmatrix}} = (1/4,\ 3/13) = \cdot43, \text{ reversible complement.}$$

14. To prove that the expression obtained for x' does satisfy the quadratic equation (9).

$$eF - fE = kk'.$$

A simple transformation leads to

$$\begin{aligned} a' &= k(Ae - aE), & A' &= k(Af - aF), \\ b' &= k(Be - bE), & B' &= k(Bf - bF). \end{aligned}$$

By reversing the two parts of the period for x and recombining, one gets the x' period; and there results:

$$\begin{aligned} kk'P' &= -f^2P + F^2q - fFR, \\ kk'q' &= e^2P - E^2q + eER, \\ kk'R' &= 2efP - 2EFq + (eF + fE)R. \end{aligned}$$

If these expressions are substituted for a, b, etc., in (9), the equation will gain the factor $-kk'$.

Or take C.F. (13) as temporary acyclical for x'; the corresponding period will be that of x, reversed, viz. $(p/P,\ q/Q)$; the equation will gain the factor $(qP - pQ)$. H. Orfeur.

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956. The L.G.O.C. deserves well of the public for having now equipped the whole of its fleet with boxes for used tickets. The reason for its complaint that less than 5% of the issued tickets are deposited in the boxes is not far to seek. If the company were for, say, three months, to keep a notice posted inside the omnibus stating that passengers must produce their tickets on demand and deposit them in the box on leaving the omnibus, and issued instructions to the conductors at the same time to see that this was done, I think it a safe prediction that the 5% would be increased by at least 45% by the end of that period. —A letter in the *Times*, 12th December, 1932. [Per Prof. E. H. Neville.]

THEORY OF EXPONENTIAL AND LOGARITHMIC FUNCTIONS AND THEIR DERIVATIVES.

BY W. MILLER.

GIVEN the equation $s=a^t$ or its equivalent $t=\log_a s$, where the base a is a specified positive number other than unity, the problem is to find first ds/dt and dt/ds and then the power series for $s(t)$ and a series for $t(s)$.

If m is a positive integer greater than 1,

$$(r^m-1)/(r-1)=r^{m-1}+r^{m-2}+\ldots+r^2+r+1 \gtreqless m$$

according as $0<r \gtreqless 1$.

Thus $r^m-1>m(r-1)$ in all cases for r positive and not unity.

Put r^{-1} for r,

$$m(r-1)>r(1-r^{-m}).$$

Hence $$r^m-1>m(r-1)>r(1-r^{-m}).$$

Let $r=s^{\frac{1}{m}}$, which is positive.

Then $$s-1>\frac{s^{\frac{1}{m}}-1}{\frac{1}{m}}>s^{\frac{1}{m}}(1-s^{-1}),\text{ where } s \text{ is not unity.}$$

If, as ϵ tends to 0, $\frac{1}{\epsilon}(s^\epsilon-1)$ tends to a limit σ which does not depend on ϵ (see proof in appendix), we may replace ϵ by ϵ/σ. It follows that

$$\lim_{\epsilon\to 0}\frac{(s^{\frac{1}{\sigma}})^\epsilon-1}{\epsilon}=\sigma\times\frac{1}{\sigma}=1.$$

Now this limit depends apparently on $s^{1/\sigma}$ alone, where σ depends on s alone. But since the limit is always 1 no matter what is the value of s, $s^{1/\sigma}$ must be some invariable number which will be denoted by "e"; that is,

$$s=e^\sigma,$$

or $$\sigma=\lim_{\epsilon\to 0}\frac{1}{\epsilon}(s^\epsilon-1)=\log_e s.$$

Hence, when $m\to\infty$, $s-1>\log_e s>1-s^{-1}$. ....(A)

If $a=e$, $e^t-1>t>1-e^{-t}$. ....(B)

Hence $1-t<e^{-t}$ or $(1-t)^{-1}>e^t$ if $t<1$.

Thus $(1-t)^{-1}>e^t>1+t$ if $t<1$,

or $$(1-t)^{-\frac{1}{t}}>e>(1+t)^{\frac{1}{t}} \text{ if } 0<t<1.$$

Giving t the succession of values $\frac{1}{2}$, $\frac{1}{4}$, ... , we obtain

$$4>e>2\tfrac{1}{4},\ 3{\cdot}16>e>2{\cdot}44,\text{ etc.}$$

In (A), replacing s by $(s+\delta s)/s$, we have

$$\frac{\delta s}{s} > \log_e(s+\delta s) - \log_e s > \frac{\delta s}{s+\delta s},$$

δs having any value provided that $s+\delta s$ is positive. Hence the continuity of $\log_e s$ can be examined and $\frac{d}{ds}\log_e s$ and $\int \frac{ds}{s}$ found.

Multiply the right-hand inequality of (B) by e^t, which is positive.

$$e^t . t > e^t - 1 > t. \qquad \text{(C)}$$

Replace t by δt and multiply by e^t,

$$e^{t+\delta t} . \delta t > e^{t+\delta t} - e^t > e^t . \delta t.$$

This exhibits the continuity of e^t and gives $\frac{d}{dt}(e^t)$ and $\int e^t\, dt$.

(C) may be written

$$e^t . t > e^t(1-e^{-t}) = e^t - 1 > t,$$

which is true for all t, but which need be considered for *positive* values only. Each of the members is then positive from 0 to t and, if t_1 is a fixed finite number greater than any t between 0 and t, we may write

$$e^{t_1} . t > e^{t_1}(1-e^{-t}) > e^t - 1 > t$$

within the interval 0 to t. Integrating from 0 to t we again obtain positive functions which are similarly graded in magnitude, and by repetition of the process we have

$$e^{t_1} . \frac{t^n}{n!} > (-)^{n-1} . e^{t_1}\left[\frac{(-t)^{n-1}}{(n-1)!} \dots + \frac{t^2}{2!} - t + 1 - e^{-t}\right]$$

$$> e^t - 1 - t - \frac{t^2}{2!} \dots - \frac{t^{n-1}}{(n-1)!}$$

$$> \frac{t^n}{n!},$$

for t positive. The extreme members contain factors t/n which become less than 1 when n is greater than t. The last of these factors tends to zero as n tends to infinity. Hence these members themselves tend to zero as n tends to infinity. It follows that

$$e^t = 1 + t + \frac{t^2}{2!} + \dots + \frac{t^n}{n!} + \dots = \exp(t),$$

for t positive or negative. Also $e = \exp(1)$, and, since $a = e^{\log_e a}$,

$$a^t = s = \exp(t \log_e a).$$

Now $\log_e s = \log_e(1+h) - \log_e(1-h)$ where $h = (s-1)/(s+1)$ and $|h| \leqslant 1$.

From the identity

$$(1+x)^{-1} = 1 - x + x^2 + \dots + (-x)^{n-2} + \frac{(-x)^{n-1}}{1+x},$$

we have by integration from 0 to h a power series for $\log_e(1+h)$, the convergency of which is easily examined.

It follows easily that

$$\log_e s = 2\left[\frac{s-1}{s+1} + \frac{1}{3}\left(\frac{s-1}{s+1}\right)^3 + \dots\right].$$

We thus have, when s is given, $\log_e s$ and, by putting $s=a$, $\log_e a$ and, by combining the results, $\log_a s$, that is, t.

Also when t is given, s can now be calculated from $\exp(t \log_e a)$.

The problem stated is thus solved.

Alternatively, if in (A) we put $1+xt$, supposed positive, for s,

$$xt > \log_e(1+xt) > xt/(1+xt).$$

Hence $t \gtrless \log_e(1+xt)^{\frac{1}{x}} \gtrless t/(1+xt)$ according as $x \gtrless 0$.

Thus $\lim\limits_{x\to 0}(1+xt)^{\frac{1}{x}} = e^t$ for positive or negative x.

Hence and from $\lim\limits_{m\to\infty} m(s^{\frac{1}{m}}-1) = \log_e s$ we can find by the Binomial theorem the expansions for e^t, e and $\log_e(1+h)$.

The Binomial theorem or a power series is not, however, involved in the investigation of the derivatives.

APPENDIX. Let $f(s, \epsilon) \equiv (s^\epsilon - 1)/\epsilon \equiv s^\epsilon f(s, -\epsilon)$ and let ϵ tend to 0.

Then $$f(s, -\epsilon) \to f(s, \epsilon). \quad \text{(i)}$$

Let $\mu \equiv p/q$, where p, q are any finite integers, q positive and p positive or negative.

$$f(w, q\epsilon) \equiv f(w, \epsilon) \,.\, \{w^{(q-1)\epsilon} + w^{(q-2)\epsilon} + \dots + w^\epsilon + 1\}/q.$$

The fraction tends to 1 since each of the q terms of the numerator tends to 1 and hence

$$f(w, \epsilon) \to f(w, q\epsilon). \quad \text{(ii)}$$

Put s^μ for w and divide by μ. Then by (ii) and (i)

$$f(s, \mu\epsilon) \to f(s, p\epsilon) \to f(s, \epsilon).$$

Let ϵ_1 and ϵ_2 be any two of the values of $\mu\epsilon$ as μ assumes all values, except 0, from 1 to -1. It follows that, by making $|\epsilon|$ sufficiently small, $|f(s, \epsilon_1) - f(s, \epsilon_2)|$ can be made as small as we please. Hence, as $f(s, \epsilon)$ varies when ϵ tends to 0, its extreme variation cannot be greater than the extreme value of

$$|f(s, \epsilon_1) - f(s, \epsilon_2)|,$$

which can be made as small as we please by making $|\epsilon|$ sufficiently small. Hence, as ϵ tends to 0, $f(s, \epsilon)$ tends to some limit σ which does not depend on ϵ.

W. MILLER.

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957. From a script. "0^{-n} is the reciprocal of 0^n and is invaluable being infinitely great." [Per Mr. C. E. Kemp.]

DETERMINATION OF THE FOCI, DIRECTRICES, AXES AND ECCENTRICITIES OF A CONIC WHOSE EQUATION IS GIVEN WITH NUMERICAL COEFFICIENTS.

By Lawrence Crawford.

1. The method given in Smith's *Conic Sections* (1910), p. 258, fails, like other methods, to correlate foci, directrices, eccentricities and axes; the following method, based on the equations there, shows how that can be done. If the conic is an ellipse, the real eccentricity goes with the real foci and the imaginary eccentricity with the imaginary foci, but if the conic is a hyperbola there is no such relation, as both values of the eccentricity are real.

2. If (α, β) is a focus, the conic

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

is, by definition, equivalent to

$$(x-\alpha)^2+(y-\beta)^2-(lx+my+n)^2=0$$

where $lx+my+n=0$ is the directrix corresponding to the focus (α, β) and the eccentricity is given by $e^2=l^2+m^2$. The following equations are then derived:

$$l^2-1=\lambda a, \quad lm=\lambda h, \quad m^2-1=\lambda b,$$
$$ln+\alpha=\lambda g, \quad mn+\beta=\lambda f, \quad n^2-\alpha^2-\beta^2=\lambda c,$$

giving
$$\lambda(a\alpha+h\beta+g)=l(l\alpha+m\beta+n),$$
$$\lambda(h\alpha+b\beta+f)=m(l\alpha+m\beta+n),$$
$$\lambda(g\alpha+f\beta+c)=n(l\alpha+m\beta+n).$$

Hence
$$\lambda^2\{(a\alpha+h\beta+g)^2-(h\alpha+b\beta+f)^2\}=(l^2-m^2)(l\alpha+m\beta+n)^2,$$
$$\lambda^2(a\alpha+h\beta+g)(h\alpha+b\beta+f)=lm(l\alpha+m\beta+n)^2,$$
$$\lambda(a\alpha^2+2h\alpha\beta+b\beta^2+2g\alpha+2f\beta+c)=(l\alpha+m\beta+n)^2,$$

but
$$l^2-m^2=\lambda(a-b) \quad \text{and} \quad lm=\lambda h.$$

Thus
$$\lambda\{(a\alpha+h\beta+g)^2-(h\alpha+b\beta+f)^2\}=(a-b)(l\alpha+m\beta+n)^2,$$
$$\lambda(a\alpha+h\beta+g)(h\alpha+b\beta+f)=h(l\alpha+m\beta+n)^2,$$
$$\lambda(a\alpha^2+2h\alpha\beta+b\beta^2+2g\alpha+2f\beta+c)=(l\alpha+m\beta+n)^2.$$

Also
$$\lambda(a+b)=l^2+m^2-2=e^2-2;$$
hence
$$e^2=2+\lambda(a+b).$$

3. To find the two sets of foci, etc., put

$$a\alpha+h\beta+g=\rho(h\alpha+b\beta+f).$$
$$\lambda(\rho^2-1)(h\alpha+b\beta+f)^2=(a-b)(l\alpha+m\beta+n)^2,$$
$$\lambda\rho(h\alpha+b\beta+f)^2=h(l\alpha+m\beta+n)^2.$$

Thus
$$(\rho^2-1)/\rho=(a-b)/h$$
or
$$h\rho^2-(a-b)\rho-h=0,$$

a quadratic equation in ρ *with roots real and different*, if the case

of the conic a circle is excluded. Take one root of this, say, $\rho=\rho_1$, then

$$\lambda\rho_1(h\alpha+b\beta+f)^2=h(l\alpha+m\beta+n)^2$$

and $\lambda(a\alpha^2+2h\alpha\beta+b\beta^2+2g\alpha+2f\beta+c)=(l\alpha+m\beta+n)^2.$

Thus $h(a\alpha^2+2h\alpha\beta+b\beta^2+2g\alpha+2f\beta+c)=\rho_1(h\alpha+b\beta+f)^2$

and $a\alpha+h\beta+g=\rho_1(h\alpha+b\beta+f)$

are two equations giving two sets of values for (α, β), that is, *two foci*. The line through this pair is $ax+hy+g=\rho_1(hx+by+f)$, *the corresponding axis*. Note that from the equation in ρ it follows that $\rho_1(a-h\rho_1)=-(h-b\rho_1)$, and hence this axis is parallel to the line $x-\rho_1y=0$.

Also, from earlier equations

$$(a\alpha+h\beta+g)/l=(h\alpha+b\beta+f)/m$$

so $l=\rho_1m$

and $\lambda h=lm=\rho_1m^2=\rho_1(\lambda b+1),$

or $\lambda(h-\rho_1b)=\rho_1.$

Now
$$e^2=2+\lambda(a+b)$$
$$=2+\frac{\rho_1(a+b)}{h-\rho_1b},$$

giving *the eccentricity for this pair of foci*.

For the directrices, from earlier equations

$$l:m:n=a\alpha+h\beta+g:h\alpha+b\beta+f:g\alpha+f\beta+c,$$

therefore corresponding to any one focus (α, β), the *corresponding directrix* can be found. The directrix is here given as the polar of the focus. Note that for this pair of foci, $l=\rho_1m$, hence the two directrices corresponding to the two foci given by $\rho=\rho_1$ are both parallel to $\rho_1x+y=0$, verifying that they are perpendicular to the corresponding axis $x-\rho_1y=\text{constant}$.

The foci, etc., are worked out in the same way for the second root, $\rho=\rho_2$, and since $\rho_1\rho_2=-1$, the axes of the conic, parallel to $x-\rho_1y=0$, $x-\rho_2y=0$ are verified to be at right angles.

4. *Example.* $2x^2-12xy-7y^2+18x+36y-27=0.$

The equation in ρ is $-6\rho^2-9\rho+6=0$, that is, $2\rho^2+3\rho-2=0$. Hence $\rho_1=\frac{1}{2}$, $\rho_2=-2$.

(i) Take $\rho=\frac{1}{2}$, then $e^2=2+\frac{1}{2}\cdot\dfrac{-5}{-6+\frac{7}{2}}=3.$

The corresponding axis is $(2x-6y+9)=\frac{1}{2}(-6x-7y+18)$, that is, $y=2x$. For the foci on this axis

$$-6(2\alpha^2-12\alpha\beta-7\beta^2+18\alpha+36\beta-27)=\tfrac{1}{2}(-6\alpha-7\beta+18)^2$$

and $\beta=2\alpha.$

Thus $\alpha=0$, $\beta=0$, and $\alpha=\frac{9}{5}$, $\beta=\frac{18}{5}$, *real foci*.

For $\alpha=0$, $\beta=0$ we have

$$l:m:n=9:18:-27,$$

hence the corresponding directrix is $x+2y=3$; and for $\alpha=\frac{9}{5}$, $\beta=\frac{18}{5}$,

$$l:m:n=-9:-18:54,$$

and the corresponding directrix is $x+2y=6$.

The other axis of the conic must be midway between these two directrices and so has the equation $2x+4y=9$.

(ii) Take $\rho=-2$, then $e^2=2+\dfrac{(-2)(-5)}{-6-14}=\frac{3}{2}$.

The corresponding axis is $2x-6y+9=-2(-6x-7y+18)$, that is, $2x+4y=9$. For the foci on this axis

$$-6(2\alpha^2-12\alpha\beta-7\beta^2+18\alpha+36\beta-27)=-2(-6\alpha-7\beta+18)^2$$

and $$2\alpha+4\beta=9.$$

Thus $\alpha=\frac{9}{10}\mp\frac{9}{5}i$, $\beta=\frac{9}{5}\pm\frac{9}{10}i$, *imaginary foci*.

For these values of α and β

$$l:m:n=\mp 9i:\pm\tfrac{9}{2}i:\tfrac{27}{2},$$

hence the corresponding directrices are $2x-y\pm 3i=0$.

Again, the first axis is midway between these two directrices, and has the equation $2x-y=0$.

5. *Special case.* The conic is a parabola, that is, $h^2=ab$; the above method holds in this case.

The equation in ρ is $h\rho^2-(a-b)\rho-h=0$; hence

$$\begin{aligned}\rho&=[a-b\pm\sqrt{\{(a-b)^2+4h^2\}}]/2h\\&=[a-b\pm(a+b)]/2h\\&=a/h \text{ or } -b/h.\end{aligned}$$

(i) $\rho_1=a/h=h/b$.

Then $h-\rho_1 b=0$ and $e^2=\infty$. The axis has the equation

$$ax+hy+g=\rho_1(hx+by+f)$$

or $$ax+hy+g=ax+hy+\frac{af}{h}.$$

Here axis and foci are at infinity.

(ii) $\rho_2=-b/h=-h/a$.

Then $h-\rho_2 b=h(a+b)/a$ and $e^2=2-\dfrac{h}{a}\cdot\dfrac{a(a+b)}{h(a+b)}=1$.

The corresponding axis has equation $ax+hy+g=\rho_2(hx+by+f)$, that is,

$$\begin{aligned}ax+hy+g&=\rho_2\frac{b}{h}\left(ax+hy+\frac{fh}{b}\right)\\&=-\frac{b}{a}\left(ax+hy+\frac{fh}{b}\right),\end{aligned}$$

or $$(a+b)(ax+hy)+ag+fh=0.$$

For foci, $(a+b)(a\alpha+h\beta)+ag+fh=0$, giving $a\alpha+h\beta=\lambda$, say.

Also $h(a\alpha^2+2h\alpha\beta+b\beta^2+2g\alpha+2f\beta+c)=-\dfrac{h}{a}(h\alpha+b\beta+f)^2$,

or $$(a\alpha+h\beta)^2+a(2g\alpha+2f\beta+c)=-\left\{\frac{b}{h}\left(a\alpha+h\beta+f\frac{h}{b}\right)\right\}^2;$$

thus $$\lambda^2+a(2g\alpha+2f\beta+c)=-\frac{b}{a}\left(\lambda^2+\frac{2fh}{b}\lambda+f^2\frac{h^2}{b^2}\right),$$

or $$a^2(2g\alpha+2f\beta+c)=-(a+b)\lambda^2-2fh\lambda-af^2,$$

and $a\alpha+h\beta=\lambda$, hence there are two simple equations for α and β, and there is one *real focus*. The directrix can be found by the general method.

6. *Example*. $4x^2+12xy+9y^2+2x+2y+2=0$.

The equation for ρ is $6\rho^2+5\rho-6=0$, hence $\rho=\frac{2}{3}$ or $-\frac{3}{2}$.

(i) $\rho_1=\frac{2}{3}$; $h-\frac{2}{3}b=0$ and $e^2=\infty$. For the axis

$$4x+6y+1=\rho_1(6x+9y+1) \text{ gives } 4x+6y+1=4x+6y+\tfrac{2}{3},$$

which is the line at infinity.

(ii) $\rho_2=-\frac{3}{2}$; $\rho_2(a+b)/(h-\rho_2 b)=-1$ and so $e^2=1$.

The axis has equation $4x+6y+1=-\frac{3}{2}(6x+9y+1)$, that is,

$$26x+39y+5=0.$$

For foci, $26\alpha+39\beta+5=0$, giving $2\alpha+3\beta=-5/13=\lambda$, say,

and $$6(4\alpha^2+12\alpha\beta+9\beta^2+2\alpha+2\beta+2)=-\tfrac{3}{2}(6\alpha+9\beta+1)^2;$$

thus $$4(\lambda^2+2\alpha+2\beta+2)+(3\lambda+1)^2=0,$$

$$8(\alpha+\beta+1)=-(13\lambda^2+6\lambda+1).$$

Hence $\alpha+\beta=-14/13$ and $2\alpha+3\beta=-5/13$, and the focus is $(-37/13, 23/13)$. For these values of α and β,

$$a\alpha+h\beta+g=\tfrac{3}{13},\quad h\alpha+b\beta+f=-\tfrac{2}{13},\quad g\alpha+f\beta+c=\tfrac{12}{13},$$

and so the equation of the directrix is $3x-2y+12=0$.

L. C.

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958. " A person (said he) had for these last five weeks often called at my door. . . . At last we met, and he told me that he was oppressed by scruples of conscience . . . that he was clerk to a very eminent trader, at whose warehouses much business consisted in packing goods in order to go abroad : that he was often tempted to take paper and packthread enough for his own use. . . . Then (replied I) . . . I would advise you Sir, to study algebra, if you are not an adept already in it : your head would get less *muddy*, and you will leave off tormenting your neighbours about paper and packthread, while we all live together in a world that is bursting with sin and sorrow. It is perhaps needless to add, that this visitor came no more."—*Anecdotes of Samuel Johnson by Hesther Lynch Piozzi*, Cambridge Miscellany Edition, 1932, p. 146. [Per Mr. F. P. White.]

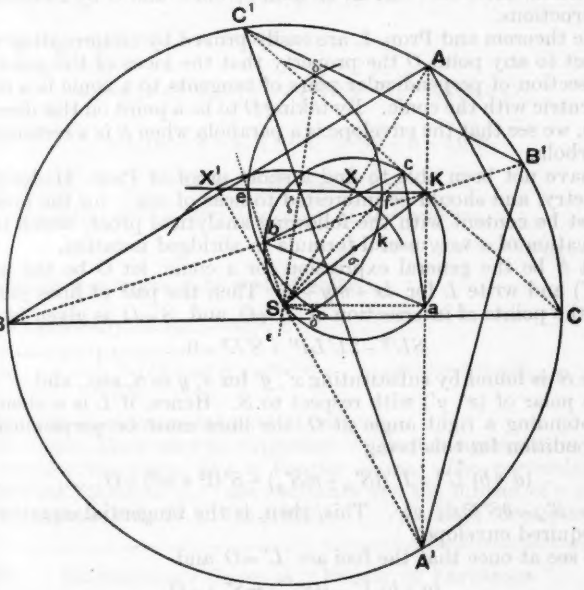
959. Ja, es ist wahr, die Mathematik ist Glatteis, aber nur für den, der sich fürchtet, Schlittschuhe unter die Füsse zu schnallen. Wenn er diese Furcht erst einmal überwunden und dann, was bekanntlich nicht allzu schwer ist, " Laufen gelernt " hat, dann kommt er nirgends sicherer und rascher zum Ziele als auf dem spiegelnden Glatteis der Mathematik.—F. Auerbach, *Lebendige Mathematik*, 1929, p. 347. [Per Prof. E. H. Neville.]

MATHEMATICAL NOTES.

1097. ***To find the triangles of which a given triangle is Brocard's second triangle.***

Let abc be the given triangle, and let σ be its circumcentre. Describe the circle abc, and find the symmedian point k of the triangle abc. Through k draw the diameter $S\sigma kK$. Then S is the circumcentre, and K the symmedian point of the required triangles. Through K draw a parallel to Sa, cutting Sb, Sc in e, f. With centres e, f, and radii ef, fe, describe arcs intersecting in δ, δ'; and on δS produced make $S\epsilon$ equal to $S\delta'$. Bisect $\delta\epsilon$ in ζ; and with centre S and radius equal to $\delta\zeta$ describe an arc cutting Ke in X, X'. Let SX, SX' meet the line aK in A, A'; and with centre S describe the circle through A, A'. Let Kb meet this circle in B, B'; and let Kc meet it in C, C'; taking care that A, B, C; A', B', C' are in opposite cyclic order to that of a, b, c.

Then ABC, $A'B'C'$ are the triangles required.



The construction depends on two theorems which can be proved geometrically. (i) The symmedian point of Brocard's second triangle of a triangle ABC lies on the diameter SK of the Brocard circle, and is nearer to K than to S. (ii) Either focus of the ellipse that has the centroid of a triangle PQR for its centre, and touches QR,

RP, PQ at their midpoints, is the symmedian point of its pedal triangle with respect to PQR.

I should be glad to know whether any simpler construction can be found.

G. WOTHERSPOON.

1098. *A generalisation of the Frégier point.*

A theorem given in Russell's *Treatise on Pure Geometry* is as follows: the envelope of a chord of a conic S which subtends a right angle at a fixed point O, not on the conic, is a conic having a focus at O.

There are other properties of this envelope which may be of interest.

I. The directrix corresponding to O is the polar of O with respect to S.

II. If O' is the other focus, the angle between the lines drawn from the centre of S to O, O' is bisected by the axes of S.

Hence, although the conic is defined as an envelope, its foci, directrices, axes, etc., can all be found from S and O by elementary constructions.

The theorem and Prop. I. are easily proved by reciprocating with respect to any point O the property that the locus of the points of intersection of perpendicular pairs of tangents to a conic is a circle concentric with the conic. By taking O to be a point on this director circle, we see that the envelope is a parabola when S is a rectangular hyperbola.

I have not been able to find a short proof of Prop. II. by pure geometry, and should be interested to hear of one; for the present I must be content with the following analytical proof, which is an application of a very useful formula in abridged notation.

Let S be the general expression for a conic, let O be the point (x', y') and write L for $lx + my + n$. Then the pair of lines joining O to the points of intersection of $L = O$ and $S = O$ is given by

$$SL'^2 - 2L'LP' + S'L^2 = 0 \text{ ....(i)}$$

where S' is found by substituting x', y' for x, y in S, etc., and $P' = 0$ is the polar of (x', y') with respect to S. Hence, if L is a chord of S subtending a right angle at O, the lines must be perpendicular, the condition for this being

$$(a+b)\, L'^2 - L'\,(lS'_x + mS'_y) + S'\,(l^2 + m^2) = O \text{ ....(ii)}$$

where $S'_x = \partial S'/\partial x'$, etc. This, then, is the tangential equation of the required envelope.

We see at once that the foci are $L' = O$ and

$$(a+b)\, L' - (lS'_x + mS'_y) = O.$$

The first equation gives the point O, and the second equation gives the point with coordinates

$$x' - \frac{S'_x}{a+b},\quad y' - \frac{S'_y}{a+b} \text{ ....(iii)}$$

These, therefore, are the coordinates of the other focus O' of the envelope, and Prop. II. can now easily be verified.

In the case when O lies on the conic S, the envelope reduces to two points, one of which is O, and the other the Frégier point of O. The above properties reduce to well-known properties of the Frégier point, the coordinates of which are given by (iii).

A. G. Walker.

1099. *Oscillating Sequences.*

Examples of oscillating sequences are frequently given in which the odd and even terms s_{2n-1} and s_{2n} follow different laws. It is interesting to obtain a single general expression for s_n.

Let $s_{2n}=u_n$ and $s_{2n-1}=v_n$, u_n and v_n being different functions of n. Then

$$s_p=\tfrac{1}{2}\{1+(-1)^p\}u_{\frac{1}{2}p}+\tfrac{1}{2}\{1+(-1)^{p+1}\}v_{\frac{1}{2}(p+1)},$$

for this is equal to u_n when $p=2n$, and v_n when $p=2n-1$.

For example, if the sequence is

$$1,\ \tfrac{1}{1},\ 2,\ \tfrac{1}{2},\ 3,\ \tfrac{1}{3},\ \ldots$$

$s_{2n}=u_n=\dfrac{1}{n}$, $s_{2n-1}=v_n=n$, hence

$$s_n=\tfrac{1}{2}\{1+(-1)^n\}\frac{1}{\frac{1}{2}n}+\tfrac{1}{2}\{1+(-1)^{n+1}\}\tfrac{1}{2}(n+1)$$

$$=\tfrac{1}{4}\{(n^2+n+4)+(-1)^{n+1}(n^2+n-4)\}/n.$$

Similarly, if $s_{3n-1}=u_n$, $s_{3n}=v_n$, $s_{3n+1}=w_n$,

$$s_n=\tfrac{1}{3}\{(1+\omega^{n+1}+\omega^{2(n+1)})u_{\frac{1}{3}(n+1)}+(1+\omega^n+\omega^{2n})v_{\frac{1}{3}n}$$
$$+(1+\omega^{n-1}+\omega^{2(n-1)})w_{\frac{1}{3}(n-1)}\},$$

where ω is a complex cube root of unity.

And generally, if $s_{kn-r}=u_n^{(r)}$,

$$s_n=\frac{1}{k}\sum_{r=0}^{k-1}(1+\omega^{n+r}+\omega^{2(n+r)}+\ldots+\omega^{(k-1)(n+r)})\,u^{(r)}_{(n+r)/k},$$

where ω is a primitive kth root of unity.

Of course, these may be expressed, without imaginaries, in terms of circular functions, like a Fourier series. The particular case where the functions $u_n^{(r)}$ are constants was the subject of a note in *Math. Gazette*, Vol. XI, p. 114 (1922). D. M. Y. Sommerville.

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960. Shakespeare's Grasp of "Frames of Reference".

Think not the king did banish thee
But thou the king . . .
Go, say I sent thee forth to purchase honour
Not that the king exiled thee.

—John of Gaunt to Bolingbroke, *Richard II*, Act I, Sc. 3, lines 279 ff. [Per Rev. S. H. Clarke.]

REVIEWS.

Contributions to the Calculus of Variations, 1931-1932. Theses submitted to the Department of Mathematics of the University of Chicago. Pp. vii, 523. 16s. 6d. 1933. (University of Chicago Press; Cambridge University Press)

Under the leadership of G. A. Bliss the calculus of variations has flourished at Chicago for the last twenty-five years. So there is plenty of experience behind this volume, and it is to be hoped that the series, of which it is the second, will be a long one. In an introduction by G. A. Bliss and L. M. Graves, the two main reasons for its appearance are stated. First, that a collection of theses (mostly written for the Ph.D. degree) turned out by a vigorous and fairly homogeneous school is valuable to specialists. Secondly, the preparation of a thesis for a journal is apt to involve so much help from the supervisor as almost to become a collaborative effort. In preparing his first substantial piece of work it is desirable for an author to be free of a larger canvas than well-meaning, but poverty-stricken, editors are eager to supply. The former is convincing, and though there is plenty to be said against the second point, the plan seems good. I hope I shall not be understood to "damn with faint praise" if I mention the extent of the bibliographies attached to some dissertations as a lower bound to the value of the volume.

The table of contents is:

1. Edge conditions for multiple integrals in the calculus of variations, by J. E. Powell (pp. 1-62);
2. The Euler-Lagrange multiplier rule for double integrals, by M. Coral (pp. 63-94);
3. The condition of Mayer for discontinuous solutions of the Lagrange problem, by R. A. Hefner (pp. 95-130);
4. A problem in the calculus of variations suggested by a problem in economics, by H. H. Pixley (pp. 131-190);
5. Functions of lines and the calculus of variations, by R. G. Sanger (pp. 191-294);
6. Sufficient conditions for a problem of Mayer in the calculus of variations, by G. A. Bliss and M. R. Hesteness (pp. 295-338);
7. Sufficient conditions for the general problem of Mayer with variable end points, by M. R. Hesteness (pp. 339-360);
8. The problem of Bolza and its accessory boundary value problem, by Kuen-Sen Hu (pp. 361-444);
9. Jacobi's condition for multiple integral problems of the calculus of variations, by A. W. Raab (pp. 445-474);
10. A history of the classical isoperimetric problem, by T. I. Porter (pp. 475-523).

These were all written for the doctor's degree except (6) and (10), the former being included because of its relation to (7), and the latter being for a master's degree. Two are of an historical nature, namely (5) and (10), while the remainder represent original work.

Functions of curves were introduced by Volterra in 1887, and the analogy between a function of a variable curve and a function of one or more real variables has been a vital source of inspiration to subsequent mathematicians. For example, it led Fréchet to make the first systematic study of what are often called "abstract spaces", while the application to quantum mechanics of functions, or operators, which make functions correspond to functions, is as dramatic as anything in Science. R. G. Sanger (5) is concerned with the first kind, which attach a real number to each one of a set of functions, curves,

surfaces or what not. His first chapter deals with the work of Volterra and his followers on the derivative of a functional. He describes the criticism made by Bliss, which shows that Volterra's original definition of the derivative is inadequate for the purposes of the calculus of variations, and the consequent modifications of the theory of the derivative, mainly due to Fischer. The second chapter describes the differential of a functional, as apart from its derivative, introduced by Fréchet, and the work of Le Stourgeon, whose relation to Fréchet is similar to that between Fischer and Volterra. The third chapter describes the application of functionals to the calculus of variations.

The classical isoperimetric problem is to find the curve with a given perimeter which encloses the greatest area. T. I. Porter (10) tells how the problem was considered by Pythagoras and Archimedes, solved heuristically at the beginning of the last century, and rigorously, according to contemporary notions, by Weierstrass. Simplified proofs have been produced by W. Blaschke and other modern mathematicians.

Apart from its elegance and historical associations, the isoperimetric problem is the oldest and simplest among the class of problems referred to in the next paragraph, with which contemporary mathematicians are intensely preoccupied. So a study of both these historical papers will be profitable even to mathematicians whose interest is mainly in the present and future. Both of them support the remarks on the concluding page of the introduction, emphasizing the value of historical accounts and analyses of subjects which are now in the limelight.

Of the remaining papers five deal with problems of the isoperimetric type, namely (2), (3), (6), (7) and (8). In its most general form a problem of this type is to minimise an integral dependent on a variable locus, or set of functions, the latter being restricted by a set of conditions (*e.g.* a set of differential equations) over and above the usual boundary conditions. Certain interrelated problems of this type, associated with the names of Lagrange, Mayer and Bolza, are the subject of the papers referred to.

The paper by J. E. Powell (1) deals with a class of problems arising out of the "corner conditions" in the classical calculus of variations. At points where the fundamental integrand fails to satisfy certain regularity conditions an extremal can have a corner, *i.e.* a discontinuity in the slope of its tangent. If $f(x, y, y')$ is the integrand, the functions $f_{y'}, f-f_{y'}y'$ are continuous even at a corner. The main object of paper (1) is to derive analogous conditions in the n-dimensional problem, the analogue of a corner being an $(n-2)$-dimensional manifold, near which the direction of the normal to a minimizing $(n-1)$-dimensional manifold is discontinuous.

The paper by A. W. Raab (9) is about n-fold integrals to be minimized with or without isoperimetric conditions. After a few preliminary remarks on the subject of edge conditions, he proceeds to the main topic, namely the differential equations associated with the second variation. He is thus led to the theory of elliptic partial differential equations, a field in which there is any amount of work to be done and which occupies a central position both in geometry and in physics.

In paper (4) H. H. Pixley considers two extensions of the classical theory of corner conditions. The first is to count as admissible, arcs which themselves have a finite number of discontinuities in addition to discontinuities of the slope of the tangent. The second refers to integrands $f(x, y, y', y'')$, where y' and y'' are allowed to be discontinuous.

Pixley gives necessary conditions for the existence of extremals, which in general are not satisfied, and shows that if an extremal does exist it will satisfy conditions analogous to the classical corner condition. He then develops quite an extensive theory of the second variation which is analogous to the theory of the Jacobi differential equation. As a contribution to mathematics this paper has an element of "class" about it. Largely because of

this I think it would have been better to have cut down the passages referring to economics. Whatever the value of the latter may prove to be eventually, at present they represent something of a speculation, and they might have been published separately in greater detail. J. H. C. WHITEHEAD.

The Theory of Ruled Surfaces. By W. L. EDGE. Pp. ix, 324. 20s. 1931. (Cambridge)

This work classifies all the projectively different types of ruled surfaces of the fourth, fifth and sixth orders in ordinary three-dimensional flat space. The complete classification of the ruled quartics was given by Cremona in 1868; and a partial classification of quintics by Schwarz in 1867. The sextics are here classified for the first time.

An important feature of the book is the introductory chapter, which will be valuable to many who are not primarily interested in the more special problem of the subsequent chapters. This gives an account of the fundamental principles and methods employed, including a clear statement of the principle of correspondence; and general theorems are obtained for ruled surfaces in space of any number of dimensions. Copious references to earlier publications are given here and throughout the book.

Chapters II to VI are concerned with the special problem of determining what types of ruled surface, of order not greater than six, are to be found in three-dimensional flat space. In the course of this investigation many interesting properties of associated curves and other loci are revealed.

One method of attacking this problem is to use the representation of the lines in three-dimensional space by the points of a quadric primal Ω in five dimensions. The generators of a ruled surface R are represented on Ω by the points of a curve C, of the same order as R; the problem is thus reduced to that of determining the types of curves of given order which are found on Ω. The properties of any ruled surface R can be derived from those of the representative curve C in relation to Ω; in particular, R is developable if and only if every tangent to C lies on Ω.

A second method is to investigate the normal ruled surfaces; a ruled surface being termed normal when it cannot be obtained as the projection of a ruled surface of the same order in space of higher dimensions. Any ruled surface which is not itself normal can be obtained as the projection of a normal ruled surface of the same order. This method can be used to classify the ruled surfaces in space of any number of dimensions, whereas the first method is applicable only to ruled surfaces in three dimensions.

These two methods are used with consummate skill; the reasoning is almost entirely geometrical, although some algebraical results are also given. The surfaces considered are classified in tables, showing their double curves and bitangent developables.

The book concludes with a note on the intersections of two curves on a ruled surface, elucidating certain difficulties in applying the principle of correspondence.

The author has made a considerable advance in the knowledge of ruled surfaces, and has rendered a valuable service by exhibiting the power of the two methods above described. T. L. W.

Great Men of Science. By P. LENARD. Pp. xix, 389. 12s. 6d. 1933. (Bell)

Interest in the history of science is so widespread that a welcome is certain for any book which deals adequately with any part of this subject. Professor Lenard has ambitiously attempted to include, in one volume of moderate size, an account of all those investigators "to whom any single piece of knowledge of first-rate importance can be traced". In the case of physics and chemistry,

and to a lesser extent, biology, his success is reasonably complete, but other parts of science are ignored ; there is no mention, for example, of such pioneers as Hugh Miller the geologist, and William Harvey the physiologist. (The omission of many outstanding mathematicians is deliberate, for reasons considered later.)

The book is written in the form of biographies, but the implied discontinuity is little apparent when the work is read as a whole. There is a steady progression from the experiments and hypotheses of pioneers like Archimedes and Galileo, to the coordinated expression of observed phenomena in the equations and " laws " enunciated by the great physicists of the nineteenth century. In addition, the author has the knack of summarizing in a few lines or pages the life history of the scientist under discussion.

In a mathematical periodical, it is appropriate to devote closer attention to the position given to mathematics by Professor Lenard. His opinions (some mathematicians may regard them as prejudices) are strongly held and unhesitatingly affirmed, and are best illustrated by quotation. He says " mathematics is, throughout scientific research, simply a tool " (p. 220); " mathematicians are thus often confused with scientific investigators " (p. 221); " what is now taught . . . under the name of science is what . . . everyone who is in the first place an investigator . . . regards as a mere tool, a subsidiary matter ; mathematical technique " (p. 246) ; " the fact that this achievement was reserved for Helmholtz, who had never studied mathematics at the University at all, shows, in a striking manner, the complete uselessness of the extensive mathematical and other courses of training at present-day universities, in which innumerable students are plagued with the most out of the way matters merely for the purpose of examination . . . whereas only few are capable of originating any kind of progress by means of mathematics and have no need to waste their time in this way " (p. 295). Bearing in mind this denial that mathematicians are scientists, it is at first sight surprising that of sixty-five scientists whose lives are given, no less than fifteen would be given a prominent place in any history of mathematics. But it is seen on examination that all, with the possible exception of Euclid, are admitted for their contributions to mechanics or to mathematical physics, not for the advances in mathematical knowledge for which they are renowned. Thus we obtain an excellent impression of the magnificent structure (mainly mathematical) of modern physics, but only fleeting glimpses of the great mathematical discoveries which have made the building a possibility.

Few misprints and errors have been noticed, only one being of importance. In an account of the work of Huygens on impact, the constancy of *vis viva* (instead of momentum) is asserted. Certain obscurities and clumsiness of expression in various places would seem to be due to a too close adherence to a direct translation of German idiom.

It would be ungenerous to conclude with criticism. The reviewer has found the book fascinating and stimulating, and, however much one may dissent from some of Professor Lenard's opinions, there can be no difference of opinion as to the skill and learning with which he has carried through his task.

C. W. G.

C. A. Bjerknes, sein Leben und seine Arbeit. By V. BJERKNES. Pp. iv, 218. RM. 8.60. Geb. RM. 9.80. 1933. (Springer, Berlin)

The independence of Norway derived from the fierce nationalism aroused by the Napoleonic wars. Norwegian patriots felt, also, that they must be independent of Copenhagen for their higher education—an object attained by the foundation of the University at Oslo in 1813. Its development was hindered by national poverty, lack of men of the highest attainments to fill academic posts, and, in science, by a natural but deadening emphasis on practical needs.

The scientific curriculum was designed to train metallurgists for work in connection with the silver mines, then the great industry of the country. As a result, pure scientists, and especially mathematicians, found their road a stony one, and were often unable to realize the hopes suggested by their own desires and abilities. The tragic history of Abel is known to every mathematician. The struggles of C. A. Bjerknes—as related by his son in this book—might easily have ended in complete failure; as it was, his opportunities for research came too late to allow him to develop his work as he might have done had he started at an earlier age.

Bjerknes was born in 1825 and entered the university at Oslo in 1844, pure mathematics being taught by Holmboe, applied by Hansteen, who was really a physicist. On completing his course, as no opportunity for scientific work presented itself, he became a mining apprentice at Kongsberg (the centre of the industry), but realized more and more that his interests were speculative rather than practical. He continued his mathematical studies, and repeatedly made application for a government grant to assist him, but it was not until 1853 that he was at length successful. His scholarship was ostensibly for research, but he was required to assist in the university teaching, with the distant hope of a lectureship to spur him on. In 1855 he obtained a travelling scholarship, and in Germany was for the first time brought into touch with modern mathematical work. Dirichlet impressed him greatly, and he was one of the three students who attended Riemann's lectures on the theory of functions. On his return, he still only held his stipend as a research student, although he shared all the mathematical teaching with Broch, and it was not until 1863 that he was appointed lecturer in applied mathematics at the age of thirty-eight! Only then had he sufficient financial stability to enable him to concentrate on the research work which had lain dormant in his mind for many years.

He had, as a student, been struck by the difficulties of the conception of action at a distance, and his work under Dirichlet had suggested to him that in hydrodynamics might be found analogies which would enable one to dispense with this concept. He turned to the solution of the problem of the motion in a fluid of a number of spheres of variable volume, and, after heavy labour, succeeded in obtaining solutions which showed that, in all essentials, the motion induced in one sphere by that of others was the same as that which would be produced by attractive or repulsive forces between them. He therefore put forward the hypothesis of an elastic fluid filling all space as an alternative to action at a distance. He felt, however, that experimental evidence was necessary to convince the scientific world of the truth of the hydrodynamical appearances suggested by his mathematical theory, and, in spite of the difficulties created by his lack of equipment and facilities for the work (he had become professor of *pure* mathematics in 1869, incidentally condemning Sylow to forty years of obscurity), he succeeded in his aim. His experiments shown at the International Electrical Congress in Paris in 1881 excited great interest, and secured the acceptance of his work by his great contemporaries.

The rest of his life was devoted to extending his theory and experiments. His age prevented him from benefiting as he might have done from Maxwell's great work, and, in the end, his devotion to hydrodynamical analogies led him into errors which he would not abandon. But his work under difficulties gives him a worthy, though minor, place among the mathematical physicists of last century. In addition, he found time to write a life of Abel, in which he preserved every accessible memory of the greatest of his mathematical fellow-countrymen.

Professor V. Bjerknes has related the life of his father with an art which grips the reader's interest, and holds it to the end. C. W. G.

David Hilbert. Gesammelte Abhandlungen. II. Algebra, Invariantentheorie, Geometrie. Pp. xviii, 453. RM. 45. 1933. (Springer, Berlin)

This is the second volume of Hilbert's collected works and contains his researches on algebra, the theory of invariants and geometry, a total of 29 papers. It also includes a short appreciation of his algebraic work by van der Waerden, a discussion of his work on the foundations of geometry by Arnold Schmidt, and last but not least, an exceedingly good reproduction of an obviously recent portrait of Hilbert.

Hilbert's *Foundations of Geometry* is such a well-known book (it is not reprinted here) that it comes rather as a surprise that he has written only three papers on geometry, and of these, two deal with the forms or shapes of curves and surfaces.

The papers on algebra have a wide range and represent many aspects of the subject, *e.g.* invariants, indeterminate equations, definite forms, theory of equations, reducibility and hypercomplex numbers. The two on indeterminate equations, "Über die diophantischen Gleichungen vom Geschlecht Null", written jointly with Hurwitz, and "Über diophantische Gleichungen", would perhaps find a more appropriate place in Vol. I containing his researches in number-theory. The first paper is one of the earliest giving any general results on Diophantine equations. It shows that the general solution in integers of $f(x, y, z)=0$ where f is a homogeneous polynomial in x, y, z of degree n and $f=0$ represents a curve of genus zero, can be deduced from the known case $n=1$ or 2. It may be mentioned that Severi in his lecture at the international mathematical congress in Zürich in 1932 points out that this result is virtually contained in a theorem given in 1870 by Max Noether.

Hilbert's contributions to the theory of invariants will be of most interest to English readers. Here is easily seen what distinguishes his work and stamps him clearly as one of the greater mathematicians; the discovery of entirely new ideas, the successful grappling with most formidable problems, the scaling of apparently inaccessible heights. Here is the fashioning of a theory, the development of the requisite generality enabling a new discipline to grow from isolated facts, the marking out of the lines of future work and the obvious influence upon future research.

Before the advent of Hilbert, algebraists were greatly interested in the explicit form of invariants and a proof that they could be rationally expressed by means of a finite number of them. Hilbert's new methods were exceedingly powerful and had the widest applications. He showed in particular that his arithmetical methods applied to algebraic problems of many types. In his resumé "Über die Theorie der algebraischen Invarianten", Hilbert says that in the history of a mathematical theory, most often three stages of development can be easily and clearly distinguished, the naïve, the formal and the critical. In the theory of algebraic invariants, Cayley and Sylvester, the earliest founders of the theory, are the representatives of the naïve period. Theirs was the joy of the first discovery of the simplest invariant constructions and the elegant applications to the solutions of the equations of degree not greater than four. Clebsch and Gordan, the inventors and perfectors of the symbolic methods, are the representatives of the second period. The critical period is represented by a number of his own theorems.

These methods were given in his paper "Über die vollen Invariantensysteme" and "Über die Theorie der algebraische Formen". These had a great influence on Hilbert's future career. Klein relates that they made him determined to call Hilbert to Göttingen at the first opportunity. He also relates that they did not appeal at first to Gordan, who said, "That is not mathematics, it is theology". Later he said, "I have convinced myself that theology also has its merits".

Only a few more papers need be mentioned. The one "Über die Irreduzibilität ganzer rationaler Funktionen mit ganzzahligen Koeffizienten" is not

a bad illustration of Hilbert's power. He proves a general theorem of which the simplest case is as follows. Let $f(x, y)$ be a polynomial in x, y, with rational integer coefficients. Then if $f(x, y)$ is irreducible in x, y, that is, cannot be expressed as a product of polynomials in x, y with integer coefficients unless one of the factors is a constant, an infinity of integer values of y exist such that the polynomial in x, $f(x, y)$ is irreducible in x. The theorem, which is of some importance, is easily suggested, but the proof was not so easily found then.

There are also some papers on definite forms, *i.e.* homogeneous algebraic forms of even degree n with real coefficients and m real variables which never assume negative values and also have non-zero discriminants. Thus every definite quadratic form with m variables can be expressed as a sum of m squares of real linear forms. So every definite binary form can be expressed as the sum of the squares of two real forms, as is obvious on factorising the form. He proves that a definite ternary biquadratic form can be expressed as the sum of three squares of real quadratic forms. But in all other cases, there exist definite forms of degree n in m variables which cannot be expressed as a sum of a finite number of squares of real forms. Hilbert shows, however, that every definite form can be expressed as the quotient of the sums of squares of real forms, and hence a decomposition into a sum of squares of real rational forms is always possible.

There can be very few mathematicians whose collected works are a more valuable contribution to mathematical literature than Hilbert's, or are more necessary for libraries and institutions as a work of reference, or to individuals as a source of knowledge and inspiration, or which should command a readier sale than his. Although all interested in mathematics must be grateful to those concerned with the production, including of course the editing, the English reader cannot help feeling that in these difficult times (and the American reader is also sure to agree), the cost, which to him is nearly seventy shillings, is rather heavy. While he will look forward to the remaining volumes, he cannot help hoping that they will be less expensive. L. J. Mordell.

Mathematical Tables. III. Minimum Decompositions into Fifth Powers. 10s. 1933. Prepared by L. E. Dickson, Professor of Mathematics in the University of Chicago, and published under the supervision of the British Association Committee for the Calculation of Mathematical Tables. (Office of the British Association, Burlington House, London)

This volume is, as stated in the Preface, the first outcome of a bequest to the British Association from Lt.-Col. A. J. C. Cunningham to assist in the production of tables connected with the theory of numbers.

The tables are of importance for Waring's problem for fifth powers, that is, every positive integer n is the sum of at most N positive fifth powers where N is independent of n. The tables suggest that $N=37$, and $223=6(2^5)+31$ in fact requires 37 fifth powers. For large values of n, probably not so many fifth powers are required; thus 15 suffice for $191{,}263 < n < 300{,}000$.

It is interesting to note that though the first tables of this kind, namely Dase's for cubes, were published more than eighty years ago, the least value of the corresponding N for large n, probably 4 or 5, has not yet been established.

The tables give the actual decomposition of all integers $< 150{,}000$ into the minimum number of fifth powers, but only the minimum number when $150{,}000 < n < 300{,}000$.

Obviously, any one requiring numerical data for Waring's problem for fifth powers will need this volume. It is conveniently arranged, and those who consult it will no doubt agree that it is a book after Col. Cunningham's own heart. L. J. Mordell.

Symbolic Logic. By C. I. Lewis and C. H. Langford. Pp. xi, 506. 21s. 1932. (The Century Company, New York and London)

This admirably designed book fills a serious gap in English logical literature. There have recently been published several excellent general introductions to logic which give prominence to modern work (L. S. Stebbing's and C. A. Mace's are particularly good), and students of the foundations of mathematics can always turn to Russell's *Introduction to Mathematical Philosophy.* But for a student of symbolic logic there was no treatise giving a connected account of modern work, some of which is very inaccessible. Even if he could read German, Hilbert and Ackermann's *Grundzüge der theoretischen Logik* would only cover part of the ground of this treatise.

The first half is written by Professor Lewis of Harvard, who is known as the constructor of the system of "strict implication" and as the author of *Mind and the World-Order* (which is in my opinion the most important philosophical book that has recently come to us from America). Four chapters expound in an admirable and witty manner the Boole-Schröder Algebra of Logic, and its application to classes, propositions and propositional functions. (Professor E. V. Huntington has published in *Mind*, 1933, a slight improvement upon Lewis's postulates). Chapter VI develops a "Logistic Calculus of Unanalysed Propositions" based upon "strict implication". In this system "p strictly implies q" is defined as "It is impossible that p should be true and q false": and the system contrasts with that of *Principia Mathematica* (based upon "material implication") in that two extra postulates are required; a "consistency postulate", that the possibility of p and q strictly implies the possibility of p; and an "existence postulate", that there exist two propositions p and q such that p strictly implies neither the truth nor the falsity of q. In Chapter VIII Professor Lewis maintains that "strict implication" corresponds to deducibility so that when and only when p strictly implies q can q be deduced from p. This involves him in the paradoxes that any proposition can be deduced from a contradiction, and that a tautology can be deduced from any proposition; and his attempts to deal with these do not seem to me very satisfactory. I am inclined to think that the notion of deducibility in any interesting sense cannot be included within the body of a logistic system, and that the necessary (or tautological) nature of the implications of such a system are implicitly shown rather than explicitly stated. This would make Lewis's system less important than Russell's. But, whatever may be concluded about these highly controversial matters, "strict implication" has helped to focus attention upon the problem of deducibility, and has stimulated interesting work at Harvard and Warsaw (described in Chapter VII and Appendix II) upon the development of truth-value systems with more than two values. These are sometimes spoken of as different "logics": I prefer to call them different "algebras" some of which are applicable to logic.

The second half of the book, written by Professor Langford of the University of Michigan, contains two chapters discussing "Postulational Technique", by which is meant what European writers are more accustomed to call the "axiomatic method". Besides the actual deductions in a system, the possibility of deduction in various types of order-system is considered. (The problem is similar to Hilbert's *Entscheidungsproblem*). There are three more general chapters treating of the interpretation of different forms of proposition, which include various philosophical remarks of great acumen. I particularly liked the concluding pages in which Professor Langford discusses the analysis of propositions as "possibilities", a line of approach that has commended itself to several philosophical logicians recently.

The reader should be warned that in Langford's part of the book (and in the Index) the word "matrix" is used in its *Principia Mathematica* sense, while in Lewis's part of the book (and in the Table of Contents) "matrix"

is used for the square table employed in considering compound propositions as truth-functions of their constituent propositions.

This book does not profess to be a treatise on the foundations of mathematics, and I found many of its remarks on this subject unsatisfactory. To state (as is done in the first chapter) that in *Principia Mathematica* "in a manner peculiarly final, the nature and basis of mathematical truth is definitely determined, and demonstrated *in extenso*" is to attribute to that terrific achievement something which certainly the authors would not claim for it. And the short treatment of the Theory of Types in the last chapter I found disappointing. This was partly because of the book's somewhat superficial analysis of classes and of general propositions, both of which are essential to a philosophy of mathematics. But these matters are only subsidiary to the main purpose of the book, which is to elaborate and discuss the systems of symbolic logic. In this aim it is eminently successful; it will probably not be superseded for some time as the standard work upon this subject.

R. B. Braithwaite.

Planetary Theory. By E. W. Brown and C. A. Shook. Pp. xii, 302. 15s. 1933. (Cambridge)

This is the most important treatise, in English, on Celestial Mechanics since the publication in 1918 of Professor H. C. Plummer's *Dynamical Astronomy.* The two volumes are, however, dissimilar in scope. Plummer's book was a comprehensive survey of the general methods in lunar and planetary theory and in cognate problems, while the book under review is confined to a detailed discussion of planetary theory, based on the method of the variation of the elements (canonic theory) and on the method in which the true longitude of the planet concerned is used as the independent variable. The greater part of the book, it may be added, is devoted to the exposition of the first method.

The second, third and fourth chapters should appeal to the pure mathematician; here are given methods for the expansion of a function (with particular reference, of course, to planetary theory). The fundamental theorem used is that of Lagrange for the expansion of a function defined by an implicit equation, and the method developed is by way of symbolic operators. The extension of Lagrange's theorem for several independent variables is fully made use of and there are theorems dealing with various functions of a Fourier series. As regards this section of the work the authors remark that "the use of formal processes is justified whenever experience shows that the results, not otherwise obtainable, are useful for the prediction of physical phenomena. Thus when calculating with an infinite series whose convergence properties are not known, one has to be guided by the results obtained; if the series appears to be converging with sufficient rapidity to yield the needful degree of accuracy, there is no choice save that of using the numerical values which it gives. We have not attempted to deal with convergence questions but have retained throughout the practical point of view".

Chapters V and VI contain an admirable account of the theory of canonic equations and its adaptation to planetary motions. Chapter VIII is devoted to an elaborate investigation on Resonance with a view to its application to the solar system. Resonance occurs, for example, in the satellite systems of Jupiter and Saturn where the mean motions of two (or more) of the satellites are in the ratio of two (or more) small integers. The resulting perturbations are then very considerable and for this reason the complete exposition of the motion of such bodies is made extremely difficult. But it is the Trojan planets, whose mean motions are practically the same as the mean motion of Jupiter, that provide the most interesting case of resonance; to the elucidation of this problem the authors devote Chapter IX. An appendix on harmonic analysis completes the book.

To the writing of this treatise, Professor Brown has brought his incom-

parable knowledge of celestial mechanics and a mathematical technique of the highest order. The authors have produced a book which will rank as the authoritative work on Planetary Theory and which will be a source of inspiration to all investigators in this field of astronomy.

Kegelschnittlehre. By W. LIETZMANN. Pp. iv, 46. RM. 1.20. 1933. Mathematisch-Physikalische Bibliothek, Reihe I, Band 79. (Teubner)

Dr. Lietzmann carries the grouping of theorems round a few fundamental properties to the logical extreme of regarding these fundamental properties as six alternative definitions of the conic. (1) The locus of the centre of a circle which passes through a fixed point and touches a fixed circle; this is essentially the bifocal definition, modified to facilitate the inclusion of the parabola. (2) The section of a right cone. (3) The perspective projection of a circle. (4) The locus of the intersection of corresponding rays of two homographic pencils. (5) The curve of the second degree. (6) The equivalent of a circle in any affine transformation.

In the course of establishing the equivalence of these various definitions, which he does completely, Dr. Lietzmann touches on many of the most important theorems in the subject. In so short a work gaps are inevitable, and he himself points out that conjugate diameters are not mentioned; this is unfortunate, since the relation of the conic to a pair of conjugate diameters provides perhaps the most direct step from the definition as a curve of the second degree to an elementary planimetric definition. But the astonishing thing is how much is included, and we are grateful to the author for an inspiring booklet, which, except for a slight meanness in the margins, has been produced most attractively, with an abundance of excellent figures. E. H. N.

Elementary Calculus. I. By C. V. DURELL and A. ROBSON. Pp. viii, 240. 4s. 6d. Without appendix of further examples. 3s. 6d. 1933. (Bell)

It is impossible to write an elementary book on the calculus to suit all tastes. But for school children just beyond the school certificate stage this book would be very suitable, because it forms an easy introduction with well-planned examples and good illustrations, while it does not lead the student far astray in its logic. In many books of this type the pill of logic bears the same relation to the whole as does the grain of aniseed to the sugar in a "gob stopper".

There are two major defects. The first is that the idea of a limit is neither explained nor defined. The reviewer believes that it is possible to explain this notion without violating logic even to young children and that without it calculus is nebulous and appears to be imposed arbitrarily from above. The other is that too much stress is laid on graphical interpretation. To teach calculus without extensive illustration from graphs would be absurd, but in this book the "gradient of a function" is defined in such a manner as to suggest that the concept is primarily geometrical. It is not made sufficiently clear that the idea of relative rate of change (called gradient in the book) is logically independent of all geometry, that the geometrical case is only the easiest application. From his own experience the reviewer knows the confusion which is caused by this mistake. To come to less important details, the fact that a straight line has a unique "slope" is made to depend on the definition of $\tan\theta$. This is an inversion of the logical order. In discussing the gradient of a curve the authors should have stressed practical methods of approximation. The practical man works with tools of a certain lack of precision. All that the mathematician does, after all, is to imagine a succession of more and more accurate instruments and finally to consider an ideal case which is not subject to the accident of any particular allowed "tolerance". This should be pointed out. In handling turning points the book is so worded as to suggest that at a point of inflexion the tangent is parallel to the x-axis.

This is a prolific weed which no teacher should allow to appear even once. The use of second-order differentials is to be deplored at this stage.

In connection with the time-rate of increase a historical note concerning Newton's definition of the calculus would have been helpful.

Some details to be commended: The proof of the derivative formula for $x^{p/q}$; the use of the change of sign of $f'(x)$ as the proper criterion for maxima and minima—in place of the too common and cumbrous use of the sign of $f''(x)$; the early introduction of the formula

$$a=v\frac{dv}{ds}=\frac{1}{2}\frac{dv^2}{ds};$$

the example in which acceleration is proportional to the cube of the displacement. In the last case, however, simple harmonic motion might well have been introduced.

The reviewer has no wide knowledge of books of this class but those he has seen have either been too advanced or have been poisonous. This book is certainly simple and errors in logic are not positive. An experienced teacher will know how to counteract the defects mentioned above and how to amplify where an idea, such as that of a growing area, is introduced abruptly. Considering everything, this book is to be recommended for use in schools as an introduction to calculus, and it might well replace some books which are still used though they were only fit to be burnt twenty-five years ago. P. J. D.

Die Methoden zur angenäherten Lösung von Eigenwertproblemen in der Elastokinetic. By K. Hohenemser. Pp. 89. RM. 10.50. 1932. Ergebnisse der Mathematik, Band I, Heft 4. (Springer)

As its title suggests this book is a statement of the many different methods for finding, by approximation, the characteristic values in such vibration problems as occur in Engineering. Although it is a thin book, particularly considering its price, it is exhaustive and to be regarded more as a work of reference than a text-book.

The theory has rightly been tied together by the use of integral equations, a simple account of which is given in the introductory chapter.

While the problems discussed and methods used are all of a practical nature the formulation is somewhat condensed and the average English engineering student would require help before he could grasp it thoroughly. Nevertheless, this book is a valuable "missing link" between Courant and Hilbert's indispensable *Methoden der mathematischen Physik* and ordinary books on numerical methods. It would be a useful book of reference in the library of an engineering department.

Attention is confined to problems in one dimension, and among the methods mentioned are those of iteration, of various minimum and maximum principles, of substitution of approximating distributions and even of graphical constructions. There are plentiful references, but the absence of an index in a book of this character is particularly unfortunate.

In spite of its price the book is to be recommended to libraries and to mathematicians who are interested in engineering, or similar, problems.

P. J. D.

Graph Book. By C. V. Durell and A. W. Siddons. Revised edition. Pp. 66, xiv. 1s. 6d. Boards 1s. 9d. Also in two parts, 10d. each. 1933. (Bell)

This book was noticed in the *Gazette* in 1929, when it was first published. Its success has warranted a reissue in revised form. The changes made are few, but they are all improvements. There is rearrangement here and there, and the examples suited for discussion are specially marked so that the pathway to generalisation can be more easily traversed. Some revision examples have been added, but the size of the book has not been altered. T. M. A. C.

(1) **Leçons sur la Composition et les Fonctions Permutables.** By V. Volterra and J. Pérès. Pp. viii, 183. 20 fr. 1924. (Gauthier-Villars)

(2) **The Volterra Integral Equation of second kind.** By H. T. Davis. Pp. 76. Paper $1.00. Cloth $1.25. 1930. (Indiana University Press)

(1) The first book under review contains another chapter of the work which is being done by the Volterra school. We may infer from the circumstance that the introduction is written by Professor Volterra, who offers his thanks to Professor Péres, that the book was made by the latter. But Péres, while piously setting out the work of his *maître*, has himself made substantial contributions.

By a " composition " is meant a marriage of two functions each of two variables by the process

$$f^*g^* = \int_x^y f(x, \xi)\, g(\xi, y)\, d\xi.$$

If $f^*g^* = g^*f^*$, the functions are said to be permutable, and the process of composition then acquires the character of ordinary algebraic multiplication. The authors proceed to follow this clue, and come successively to composition powers, positive, negative and fractional, to series, functions, differentiation, and integration, all in analogy with the corresponding ideas in ordinary algebra and calculus.

The whole is constructive in character. Volterra is always content to have the conditions sufficient for his steps. He does not usually wait to search for necessary conditions, but goes on with fresh ideas.

There are difficulties. One is that there is not in general a function corresponding to unity, that is, a function f for which $f^*g^* = g$, for all the g's considered. The deficiency is made good by the device of a fictitious function 1^* with the required property. This fictitious function may be said to be justified of its works, but though the authors return to it from time to time and insert footnotes about it, the reader will probably retain his feeling of uneasiness. He may derive some comfort from the thought that composition is not altogether a matter of integration, although that is its origin. The finest part of the book is perhaps contained in chapters 3 and 4. With any function may be associated an aggregate of functions all of which are mutually permutable, and therefore form a fit subject for the theory. First the problem of finding the complete set permutable with a given function and then that of finding the group of transformations for which the permutability of any such set is conserved, are solved completely and in a manner masterly in the extreme. It will be seen that the solving of an equation in which processes of composition replace those of algebra gives the solution of an integral equation of a more or less complex nature. Indeed the same may be said for any inverse process. Thus the method affords the solution of a wide range of functional equations, some of very subtle complexity. The suggestiveness of this work is even greater than the work itself. It will be noted that all that is said is confined to integration of the Volterra type and to permutable functions at that. Beyond these boundaries lies new country.

(2) Professor Davis's little book is reminiscent of a Cambridge tract, and its quality makes it fully worthy of that comparison. It is " Contribution No. 52, Waterman Institute for Scientific Research ". It is a scholarly and well digested report on the subject of its title up to the date of its publication. It is rather more than that, since it gives in a readable form samples of most of the important theorems and makes an admirable introduction.

H. B. Heywood.

Elementary Applied Aerodynamics. By T. G. Whitlock. Pp. viii, 240. 12s. 6d. 1931. (Clarendon Press)

This is an attempt to provide an elementary textbook on Aerodynamics in

a form suitable for students of the subject whose mathematical equipment does not stretch beyond a little algebra, trigonometry, the meaning of a differential coefficient, and simple mechanical principles.

The consequence is that while such chapters as those on Air Flow and on Stability of the aeroplane are excessively simplified and primarily descriptive in character, there is still a great part of the more technical aspects of applied aerodynamics that can receive adequate treatment. Within the limits set by these restrictions the author treats problems of Parasitic Drag, Aeroplane Performance, its equilibrium, Control and Capacity for Manoeuvring, problems of Scale Effect and Airscrew Theory. The book is well supplied with practical formulae, and numerous examples illustrate the text. Altogether a successful attempt at an exceedingly difficult task. H. L.

Einführung in die höhere Mathematik. III. By H. VON MANGOLDT. Sixth edition, revised by K. KNOPP. Pp. xvi, 618. Geh. RM. 15. Geb. RM. 16.80. 1933. (Hirzel, Leipzig)

The final volume of the new edition of von Mangoldt's book is devoted to the integral calculus. Like the preceding volumes, the book is so written as to be of considerable assistance to the student who is working alone, but it is not intended as a beginner's textbook. The order and arrangement, while logical and indeed natural, are not those best adapted for the comprehension of the material by the immature pupil, who would lose interest completely if compelled to work through the first two hundred pages on the indefinite and definite (Riemann) integral before coming to simple applications. The value of these two chapters lies in their power to clarify and systematise knowledge previously gained in a first course on calculus.

The third chapter deals with the measure of the area under a curve, and also with the Peano-Jordan measure. This is emphatically a wise inclusion, for Professor Knopp has always in mind the student who is going further, and the Peano-Jordan measure, obsolete though it be, is still very useful as a means of introducing the Lebesgue theory. But we protest against Professor Knopp's suggestion that for the sake of symmetry this measure should be called Riemannian; elsewhere he has shown himself a stickler for historic accuracy in nomenclature, and there is no need for him to depart from his own standard in order to add to the already embarrassing richness of things Riemannian.

Chapter IV contains a careful analysis of the notion of length and introduces curvilinear integrals. Chapter V deals with Cauchy's theorem and its immediate consequences. Chapters VI, VII and VIII discuss multiple integrals, their applications, and the theorems of Green and Stokes. Here the author is wisely content with rigorous proofs of theorems of reasonable generality; footnotes frequently indicate the possibility of refinement of hypotheses. The reader who is attacking these subjects for the first time should be grateful for a discipline neither too lax nor too severe. Even so, he may find these chapters hard going; unfortunately, vector symbols are used in Chapter VIII only half-heartedly as a convenient shorthand for complicated Cartesian expressions, and so lack vitality.

Chapter IX, on infinite integrals and their applications to the Gamma function and to Fourier series, is an excellent piece of work, almost as good as anything in de la Vallée Poussin: nor is the last chapter an anti-climax, for in 80 pages the elementary existence theorems and methods of integration of ordinary differential equations are treated with a careful elegance wholly admirable.

Professor Knopp has given to this revision much time which he might otherwise have spent in pursuing his own valuable researches. But it has not been time wasted; he has made an antiquated though useful textbook into an up-to-date treatise, sound, comprehensive and nearly always readable, on the elements of mathematical analysis. T. A. A. B.

Storia delle matematiche. II. I secoli XVI e XVII. By GINO LORIA. Pp. 595. Lire 23. 1931. (Sten, Torino)

The first volume of this work was reviewed in the *Mathematical Gazette* for July 1930 (p. 172). The author's first intention was to complete the history, up to the end of the nineteenth century, in two volumes, but he has wisely extended his plan, and this second volume covers no more than the sixteenth and seventeenth centuries. It is an admirably clear, eminently readable and very entertaining account of the first beginnings of modern mathematics.

In algebra the great triumph of the sixteenth century was the solution of cubic and biquadratic equations, the glory of which belongs almost entirely to Italy. About the same time came the initial steps in the theory of equations of any degree; for this, however, the old notation—Professor Loria coins the happy phrase "syncopated algebra"—was plainly inadequate, and so we get the gradual evolution of a systematic symbolism. Thus only became possible the application of algebra to geometry, associated with the name of Descartes. This in turn led to a revival of interest in pure geometry, in the hands of such men as Desargues and Pascal. And then, after the "indivisibles" of Cavalieri, the maxima and minima of Fermat, the "method of tangents" of Roberval and de Sluse, and the lectures of Barrow, we come to Newton and Leibniz and the differential and integral calculus.

It is a wonderful story, and Professor Loria tells it well, entering as far as is necessary into the biographies of the chief actors and their often acrimonious disputes, but never losing sight of his main purpose, the gradual development of the subject.

An interesting chapter is devoted to the influence of the Renaissance in recalling attention to Greek mathematics, leading to the first printed editions of Euclid, Archimedes, Apollonius, Diophantus, and then to attempted "divinations" of their lost works. Professor Loria also describes the development of methods of publication, from the scientific correspondence centred round a particular man, such as Mersenne in France or Oldenburg in England, to scientific academies and scientific periodicals.

It is regrettable that this book should afford no exception to the rule that Italians cannot spell English personal and place-names: "Torpeley" for 'Torporley", "Teuby" for "Tenby". A "Tommaso Backer" is mentioned twice, with an explicit identification, but on p. 467 he should be Brancker, and on p. 471 Baker. Also it is surprising to see John Caswell described as "vice-rettore" of the University of Oxford; he was Vice-principal of Hart Hall.

So good a book is worthy of better paper and typography; but that presumably would have meant a much higher price. F. P. W.

Algebra for Schools. By H. ABSON. Pp. viii, 397. 4s. 6d. With answers, 5s. In parts: Part I, 1s. 6d.; Part II, 1s. 9d.; Part III, 2s. Answers to the three parts, 9d. 1933. (University Tutorial Press)

This might well be described as "Algebra Training, Vol. I". The author believes that much is to be said for the older methods of teaching, and his book faithfully reflects this creed.

It begins by exhaustively disposing of the four rules; all the letters of the alphabet perform every possible evolution; they are added, subtracted, multiplied, divided, and one seems to breathe again the atmosphere of the parade ground. The illusion is strengthened by occasional "words of command", as, for instance, "In subtraction the 'from' line goes on top".

By this means, we are told, the pupil (one had almost written "the recruit") sees that he is progressing along the right lines. He could hardly be blamed, however, for certain misgivings on meeting, in Chap. 2, "Addition", with the amazing statement that "minus signs may occur". In short, good

material suffers from a mechanically formal treatment which frequently savours of the illogical. The difficulty of introducing the minus sign before giving an adequate notion of direction has been already indicated. One finds an even more glaring example in the author's treatment of equations. Here, from a comprehensive discussion of formulae, their creation and transformation, is evolved an apparatus. Thus, terms may be transferred provided their signs are changed; and factors may be transferred provided " × " is changed to " ÷ ", and *vice versa*. An example on p. 90, " Shew that the rule for transferring terms applies from R.H.S. to L.H.S.", sufficiently indicates the attitude. We live indeed in the age of the machine!

Only after this stern novitiate does the pupil make the acquaintance of the simple equation as such, and this must pass through all the stages of brackets, fractional terms and coefficients, before it is used for even the simplest problem.

Again, an intimate knowledge of surds and of rationalisation is apparently an essential foundation stone for the quadratic equation, which in its turn must be explored to its innermost graphical and theoretical depths before it can be applied to any problem.

Such thoroughness can scarcely be maintained in a single volume whose purview is the School Certificate, and the notice afforded the more advanced subjects, progressions, variation, indices, gradient—this last with a studied avoidance of the notion of a limit—seems perfunctory by comparison.

As a compendium of algebraic information, and of examples, numerous and instructive, the teacher will find the book useful. But he will feel necessary a good deal of dissection and rearrangement. In particular, he will probably place p. 162, where the symbol " √ " is defined, before p. 84, where it is used for the first time.

E. L.

Exercises in Arithmetic for Middle and Upper Forms. By R. WALKER. Pp. 328. With answers, 3s. 6d. Without answers, 2s. 9d. 1932. (Harrap)

The arithmetic textbook, to the majority of boys, is probably little more than a collection of exercises, while, to the enterprising teacher, it must always be the servant and never the master. So a good case can be urged for the volume which professes to be merely a class-book of examples, and in no sense usurps the place of oral instruction.

The present work pleads such a case most eloquently. It consists of sets of formal graded exercises, for the most part in chapters according to subject, the space usually devoted to exposition being occupied by hints on methods and model examples worked out in full detail. The advantages of such an arrangement are evident. It offers the pupil assistance of the practical kind which most appeals to him, for who has not heard the cry " Will you please leave that one on the board ? " Moreover, it provides, in its models, a standard of neatness and orderly layout at which he can aim. The book thus becomes a valued tool which he can use.

For Mr. Walker's methods one has nothing but praise. The treatment of decimals is excellent; the use, in the early stages, of squared paper, and of a " *u* " superimposed, in place of the decimal point are particularly attractive. Contracted methods receive ample attention, and decimals are thereby made available for a wide variety of problems. The principle of proportionality is given splendid and sustained emphasis, and its power in questions, for example, of profit and loss, and stocks and shares, is made full use of. Graphical, equational and " formula " methods, too, find an adequate place.

The examples themselves are varied, practical and well graded. Easy mental questions are numerous, and an " intelligent forecast of the result " is commendably insisted on, whenever possible. The book is beautifully printed, and covers the work for School Certificate, with an extra helping of commerce and mensuration. Logarithm tables are provided, and a collection of formulae makes a useful feature.

E. L.

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